Homework Assignment 1

Instructions. Submit your answers as a PDF on Webcourses by **Wednesday, September 16, at 5 p.m. EST**. I encourage you to work in groups and discuss the problems with one another. You may also use AI tools, but remember that AI will not be available during exams. Use these problems to check your own understanding of the material. Your grade is based on participation, not correctness. You must submit your answers to receive credit for this assignment.

Question 1 Building with NAND gates

A two-input NAND gate computes

$$\mathrm{NAND}(A, B) = \neg(A \wedge B).$$

Express each operation below using only two-input NAND operations, and state the number of NAND gates required. Verify each expression using both Boolean identities or a truth table.

(a) NOT: $\neg A$.

(b) AND: $A \wedge B$.

(c) OR: $A \vee B$.

(d) XOR: $A \oplus B$, true exactly when A and B differ.

You may connect the same signal to both inputs of a gate and reuse a gate's output. Use as few gates as you can; a proof of minimality is not required.

Question 2 Proving an arithmetic equality

Prove that $1 + 2 = 2 + 1$ using only the numeral definitions

$$1 := S(0), \qquad 2 := S(1), \qquad 3 := S(2)$$

and the recursive rules for addition used in lecture:

$$0 + m := m, \qquad S(n) + m := S(n + m).$$

Here S is the successor operation. Justify each step. Do not assume commutativity of addition or simply cite the two numerical sums.

For Questions 3–7, you may use the usual arithmetic laws. The symbol $\mathbb{Z}$ denotes the integers. An integer is even if it equals $2k$ and odd if it equals $2k+1$ for some $k \in \mathbb{Z}$; every integer is exactly one of even or odd. Write $a \mid b$ if $b = ak$ for some $k \in \mathbb{Z}$ (alternatively, we say a divides b)

Question 3 Direct proofs

Prove each statement directly from the definitions.

(a) If a and b are odd integers, then $a^2 + b^2$ is even.

(b) For all integers a, b, c, r, s, if $a \mid b$ and $a \mid c$, then $a \mid (rb + sc)$.

Question 4 Proof by contrapositive

Consider the claim: for all integers m and n, if mn is odd, then both m and n are odd.

(a) State the contrapositive, simplifying the negation of "both m and n are odd."

(b) Prove the claim by proving its contrapositive.

Question 5 Proof by contradiction

Prove each statement by contradiction. Clearly state your assumption and identify the contradiction you obtain.

(a) There is no greatest odd integer.

(b) No integer $d > 1$ divides two consecutive integers. That is, there are no integers d, n with $d > 1$ such that both $d \mid n$ and $d \mid (n + 1)$.

Question 6 Proof by cases

Prove that for every integer n,

$$3 \mid n(n + 1)(n + 2).$$

In words, the product of any three consecutive integers is divisible by 3.

Hint. You may use the fact that every integer has exactly one of the forms $3k$, $3k + 1$, or $3k + 2$, where $k \in \mathbb{Z}$. Consider these three cases and explain why they cover every integer.

Question 7 Nested quantifiers

Consider the following statements, where every variable ranges over the integers:

A. $\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, (x + y = 0)$,

B. $\exists y \in \mathbb{Z}\, \forall x \in \mathbb{Z}\, (x + y = 0)$,

C. $\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, (2y = x)$,

D. $\exists x \in \mathbb{Z}\, \forall y \in \mathbb{Z}\, (xy = y)$.

(a) Translate each statement into English and determine whether it is true or false. Prove every true statement and justify every false one with a counterexample or an argument ruling out all proposed witnesses.

(b) Negate statements A and B, moving negations all the way inward to the equations.

(c) Explain precisely how the allowed dependence of y on x differs between A and B.

Question 8 Translating English into logic

Let the domain D be all people. Use the predicates

$$\begin{aligned} S(x) &: \ x \text{ is a student in this course,} \\ T(x) &: \ x \text{ submitted Homework Assignment 1,} \\ H(x, y) &: \ x \text{ helped } y \text{ with this assignment.} \end{aligned}$$

For each sentence, (i) translate it into a formula using these predicates, equality, logical connectives, and quantifiers; and (ii) write its negation, with every negation applied directly to a predicate or equality. Every variable ranges over D.

(a) Every student in this course submitted the assignment.

(b) Some student in this course helped every other student in this course.

(c) Every student in this course was helped by a different student in this course.

“Other” and “different” mean a distinct person. Pay attention to the order of the arguments in H.