Lecture 2, Part 2: Peano Axioms

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3 The Natural Numbers

In the first part of this lecture, we defined Boolean operations and studied what we could build from them. We now turn to a familiar number system: the natural numbers $0, 1, 2, \ldots$. Our goal is to prove that $1 + 2 = 3$ from first principles. To do this, we must specify our starting assumptions and give precise meanings to the symbols 1, 2, 3, and $+$. It is helpful to think of this process as writing a computer program: before using an operation such as $+$ or $\times$, we must specify what kinds of inputs it accepts and how its output is determined. Familiarity with these symbols does not replace the need to define them.

3.1 Zero and the Successor Operation

An **axiom** is a statement we accept as a starting assumption. We will use the five **Peano axioms**, in the order given in Terence Tao's *Analysis I*.[1]

Before stating the axioms, we introduce a distinguished object called 0 and a **successor operation**, denoted by S. For an object a, we write $S(a)$ for its successor. At this stage, we have introduced these symbols without specifying their properties. That is what the axioms will do.

The intended idea is that applying S takes us one step forward in counting. Starting from 0, repeated applications of S should give us the natural numbers:

$$0, \qquad S(0), \qquad S(S(0)), \qquad S(S(S(0))), \qquad \ldots.$$

The Peano axioms make this idea precise by specifying the rules that 0 and the successor operation must satisfy.

We have not yet defined addition, so we cannot define $S(n)$ by writing $n + 1$. Instead, we will use the successor operation to define addition later. The pictures below use familiar numbers as labels to illustrate possible successor operations; we will define the numerals formally afterward.

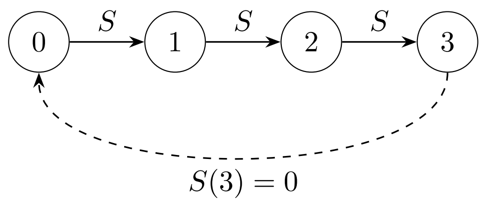
[1] Reference: Terence Tao, *Analysis I*, Section 2.1 (Axioms 2.1–2.5) and Section 2.2 (Definition 2.2.1). See the author's book page.

3.2 The Five Peano Axioms

Axiom 1. A starting point. The object 0 is a natural number.

Axiom 2. We can take another step. If a is a natural number, then its successor $S(a)$ is a natural number.
Axiom 2 allows us to keep applying the successor operation, but it does not prevent us from looping back to zero. The next axiom rules out this possibility.

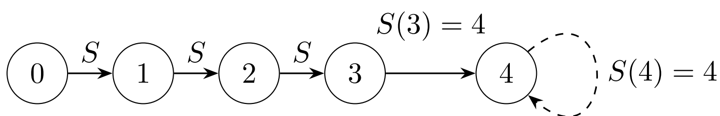
Axiom 3. No step leads to zero. For every natural number a, we have $S(a) \neq 0$.



Ruled out by Axiom 3: the dashed arrow leads back to 0.

Even with Axiom 3, two different numbers could still have the same successor. For example, we could have $S(3) = 4$ and $S(4) = 4$. The next axiom prevents this behavior.

Axiom 4. Steps from different numbers cannot merge. If a and b are natural numbers and $a \neq b$, then $S(a) \neq S(b)$. Equivalently, if $S(a) = S(b)$, then $a = b$.



Ruled out by Axiom 4: $3 \neq 4$, but $S(3) = S(4) = 4$.

Axiom 5. Induction (high-level idea). If a property holds at 0, and it passes from any natural number to its successor, then it holds for every natural number.

Intuitively, Axiom 5 says that starting at 0 and repeatedly applying S is the only way to obtain natural numbers. Axioms 1–4 give us a chain starting at 0, but they do not rule out additional, disconnected chains. The two-chain diagram on page 10 shows one such candidate system; every arrow specifies an application of S.

In the two-chain diagram, starting from 0 never reaches the chain containing 0.5. Axiom 5 rules out these extra objects: every natural number must be reached from 0 after finitely many successor steps, as in the single-chain diagram. We will study induction as a proof technique later.

Allowed by Axioms 1–4: two separate chains

$$0 \xrightarrow{S} 1 \xrightarrow{S} 2 \xrightarrow{S} 3 \xrightarrow{S} \cdots$$

$$0.5 \xrightarrow{S} 1.5 \xrightarrow{S} 2.5 \xrightarrow{S} 3.5 \xrightarrow{S} \cdots$$

With Axiom 5: every object is reached from 0

$$0 \xrightarrow{S} 1 \xrightarrow{S} 2 \xrightarrow{S} 3 \xrightarrow{S} \cdots$$

3.3 Giving Names to Numbers

We now introduce the familiar names:

$$1 := S(0), \qquad 2 := S(1), \qquad 3 := S(2).$$

The notation := means "is defined to be." Thus 2 is a name for $S(S(0))$, and 3 is a name for $S(S(S(0)))$. Axioms 1 and 2 ensure that these are natural numbers. The first few steps look like

$$0 \xrightarrow{S} 1 \xrightarrow{S} 2 \xrightarrow{S} 3 \xrightarrow{S} \cdots .$$

3.4 Defining Addition

We have names for numbers and an operation that takes one step. We can now define addition by repeating that operation. Following Tao's convention, we fix a natural number m and define $n + m$ by building up the *first* input n:

$$\begin{aligned} 0 + m &:= m, && \text{(starting rule)} \\ S(n) + m &:= S(n + m). && \text{(successor rule)} \end{aligned}$$

The first rule supplies a starting value. The second tells us how to obtain the next sum from the preceding one. This is a **recursive definition**. Intuitively, $n + m$ starts at m and takes n successor steps.

These are defining rules for addition. Familiar laws such as $a + b = b + a$ would need their own proofs; we will not need them for the calculation below.

3.5 Proving $1 + 2 = 3$

Proposition. $1 + 2 = 3$.

Proof. Using the definitions above, we obtain

$$\begin{aligned}
1 + 2 &= S(0) + 2 && \text{since } 1 := S(0) \\
&= S(0 + 2) && \text{by the successor rule for addition} \\
&= S(2) && \text{by the starting rule } 0 + 2 = 2 \\
&= 3 && \text{since } 3 := S(2).
\end{aligned}$$

□

Every equality has a stated reason. This is the habit we want to carry into more complicated proofs: identify the assumptions and definitions, then justify each step that leads to the claim.