

Scribe Notes: Introduction to Graph Theory, Vertex Degrees, and Eulerian Circuits

Course: COT 3100C — Honors Introduction to Discrete Structures

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Lecturer: Karthik Gajulapalli

Scribe: Mateusz Jelinski

1. Abstract

In this lecture, we formally introduce graph theory. We define graphs, vertices, edges, and vertex degrees, and prove fundamental structural properties: the Handshake Lemma, the parity of odd-degree vertices, and the guarantee that every simple graph has at least two vertices of identical degree (via the Pigeonhole Principle).

We then investigate the historic **Seven Bridges of Königsberg** problem and formalize **Eulerian Circuits**. We state and prove **Euler's Theorem**: a connected graph contains an Eulerian circuit if and only if every vertex has an even degree. For the reverse direction ($\Leftarrow$), we study **Hierholzer's Algorithm**, which constructively builds an Eulerian tour by iteratively splicing sub-cycles. Finally, we apply this theory to show why the Königsberg bridge problem is impossible.

2. Fundamentals of Graphs

2.1 Graph Definition

An **undirected graph** $G = (V, E)$ consists of: * A set of **vertices** (or nodes) V . * A set of **edges** E , where each edge $e \in E$ is an unordered pair $\{u, v\}$ of distinct vertices in V .

A graph is **connected** if there is a path between every pair of vertices.

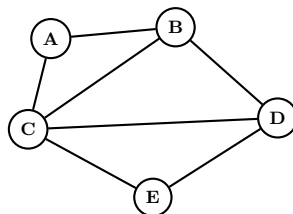


Figure 1: Example Connected graph

2.2 Degree of a Vertex

The **degree** of a vertex v , denoted $\deg(v)$, is the number of edges incident to v :

$$\deg(v) = |\{e \in E \mid v \in e\}|$$

For the graph in Figure 1, for example:

$$\deg(A) = 2, \quad \deg(B) = 3, \quad \deg(E) = 2$$

3. Fundamental Lemmas on Vertex Degrees

3.1 The Handshake Lemma

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof (Double Counting) Each edge $e = \{u, v\} \in E$ connects exactly two endpoints. When we sum the degrees of all vertices $\sum_{v \in V} \deg(v)$, each edge contributes +1 to $\deg(u)$ and +1 to $\deg(v)$. Thus, each edge is counted exactly twice. ■

3.2 Even Number of Odd-Degree Vertices

Theorem: *In any graph $G = (V, E)$, the number of vertices with odd degree is even.*

Proof Partition the vertex set V into two disjoint subsets: $V_{\text{even}} = \{v \in V \mid \deg(v) \text{ is even}\}$ and $V_{\text{odd}} = \{v \in V \mid \deg(v) \text{ is odd}\}$

By the Handshake Lemma:

$$\sum_{v \in V_{\text{even}}} \deg(v) + \sum_{v \in V_{\text{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E|$$

- The right-hand side $2|E|$ is an even integer.
- The sum $\sum_{v \in V_{\text{even}}} \deg(v)$ is a sum of even integers, which is even.

Subtracting an even number from an even number yields an even number:

$$\sum_{v \in V_{\text{odd}}} \deg(v) = 2|E| - \sum_{v \in V_{\text{even}}} \deg(v) = \text{even}$$

Because the sum of an odd number of odd integers is always odd, $\sum_{v \in V_{\text{odd}}} \deg(v)$ can only be even if $|V_{\text{odd}}|$ is even. Thus, there must be an even number of vertices of odd degree. ■

3.3 Identical Degree Property

Theorem: Any simple graph with $n \geq 2$ vertices contains at least two vertices with the same degree.

Proof (Proof by Contradiction)

1. **Assume the opposite:** Suppose for contradiction that no two vertices have the same degree (i.e., all n vertices have strictly distinct degrees).
2. **List the possible degrees:** In a simple graph with n vertices, a vertex can be connected to at minimum 0 others and at maximum $n - 1$ others:

$$\text{Possible degrees} = \{0, 1, 2, \dots, n - 1\} \quad (n \text{ possible values})$$

3. **Every degree must be used:** Since there are n vertices and n possible degree values, every value from 0 to $n - 1$ must appear exactly once.
4. **The Contradiction:**
 - A vertex with degree 0 is isolated and connected to nobody.
 - A vertex with degree $n - 1$ is connected to every other vertex in the graph (including the degree 0 vertex!).
 - A single graph cannot contain both a vertex connected to everyone and a vertex connected to no one.

Contradiction! Therefore, our initial assumption was false, and at least two vertices must share a degree. ■

4. Eulerian Circuits and Euler's Theorem

4.1 Definition

- An **Eulerian Circuit** (or **Euler Tour**) is a closed walk that begins and ends at the same vertex and traverses every edge of the graph **exactly once**.

4.2 Euler's Theorem

Theorem: A connected graph $G = (V, E)$ has an Eulerian circuit if and only if every vertex in V has an even degree.

$$\exists \text{ Euler Circuit in } G \iff \forall v \in V, \deg(v) \text{ is even}$$

4.3 Proof: Direction 1 ($\Rightarrow$)

Claim: If G contains an Eulerian circuit, then every vertex has even degree.

Proof: Suppose C is an Eulerian circuit in G . Consider any vertex $v \in V$:
* Every time the circuit visits v , it must enter along an edge and leave along a distinct edge that has not yet been used.
* Each pass through v consumes exactly 2 unused edges incident to v .
* For the starting/ending vertex v_0 , the circuit leaves along an initial edge and ultimately returns along a final edge, also pairing up as $+2$.

Because C uses every edge incident to v exactly once, the edges incident to v are completely partitioned into entry/exit pairs:

$$\deg(v) = 2 \times (\text{number of times } C \text{ passes through } v)$$

Thus, $\deg(v)$ is even for all $v \in V$. ■

4.4 Proof: Direction 2 ($\Leftarrow$) and Hierholzer's Algorithm

Claim: If G is connected and every vertex has an even degree, then G contains an Eulerian circuit.

Proof (Constructive): We prove this by presenting **Hierholzer's Algorithm**, which is guaranteed to construct an Euler tour.

Algorithm Steps:

1. **Find an Initial Cycle R_1 :**
 - Pick an arbitrary starting vertex $v_0 \in V$.
 - Follow unused edges greedily. Since every vertex has even degree, whenever the trail enters a vertex $u \neq v_0$, there is always an unused edge available to exit.
 - The process only terminates when it returns to v_0 and finds no unused edges leaving v_0 . This forms a closed cycle R_1 .
2. **Iterative Augmentation:**
 - If R_1 contains all edges in E , we are done.
 - Otherwise, because G is connected, there exists at least one vertex u along our current tour that has remaining unused incident edges.
 - In the subgraph of unused edges, all vertices still have even degrees (since R_1 removed exactly an even number of edges at every node).
 - Starting from u , trace a new cycle R_2 using only unused edges until returning to u .
3. **Splicing:**
 - Splice R_2 into R_1 at vertex u : traverse R_1 up to u , take the complete cycle R_2 back to u , and then finish the rest of R_1 .

- Repeat steps 2 and 3 until no unused edges remain.

Because E is finite and each iteration incorporates a non-empty set of edges, the algorithm must terminate, yielding a single closed walk traversing all edges of G exactly once. ■

4.5 Hierholzer's Algorithm: Worked Class Example

Consider the 6-vertex graph $G = (V, E)$ with vertices $\{A, B, C, D, E, F\}$ and the following adjacency: * $\deg(A) = 4$, $\deg(B) = 4$, $\deg(C) = 4$, $\deg(D) = 2$, $\deg(E) = 4$, $\deg(F) = 4$.

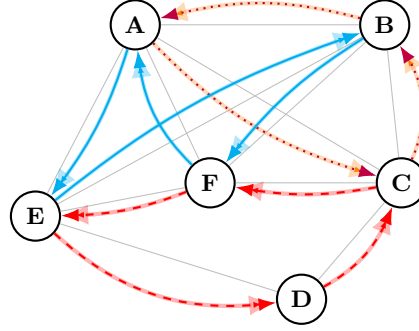


Figure 2: Traversal visualization on G . The 3 sub-cycles used by the algorithm are highlighted: Red (R_1), Purple (R_2), and Blue (R_3).

Step-by-Step Construction:

1. First Cycle (R_1):

- Start at E . Walk: $E \rightarrow D \rightarrow C \rightarrow F \rightarrow E$.
- Current Tour:

$$R_1 = [E, D, C, F, E]$$

2. Second Cycle (R_2):

- Vertex C is on R_1 and has remaining unused edges.
- Start at C . Walk: $C \rightarrow B \rightarrow A \rightarrow C$.
- Splicing R_2 into R_1 at node C :

$$R_{\text{cur}} = [E, D, C, B, A, C, F, E]$$

3. Third Cycle (R_3):

- Vertex B is on the current tour and has remaining unused edges.

- Start at B . Walk: $B \rightarrow F \rightarrow A \rightarrow E \rightarrow B$.
- Splicing R_3 into R_{cur} at node B :

$$\text{Final Tour} = [E, D, C, \mathbf{B}, \mathbf{F}, \mathbf{A}, \mathbf{E}, \mathbf{B}, A, C, F, E]$$

Every edge has been traversed exactly once, starting and ending at E .

5. The Seven Bridges of Königsberg

5.1 Historical Context

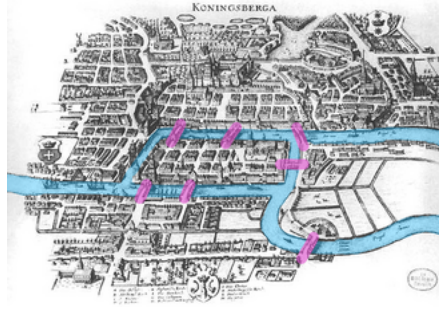


Figure 3: The Seven Bridges of Königsberg

In Königsberg, Prussia, the Pregel River divided the city into four landmasses: * North Bank (NB) * South Bank (SB) * Island 1 (I_1) * Island 2 (I_2)

Seven bridges connected these landmasses: * 2 bridges between NB and I_1 * 2 bridges between SB and I_1 * 1 bridge between NB and I_2 * 1 bridge between SB and I_2 * 1 bridge between I_1 and I_2

The Question: Can someone take a stroll through the city, crossing every bridge exactly once and returning to the starting point?

5.2 Graph Abstraction & Resolution

We model each landmass as a vertex and each bridge as an edge. To avoid a multigraph (parallel edges), we place a node on each bridge (B_1, B_2, B_3, B_4), which preserves the parity of the degree of the landmasses:

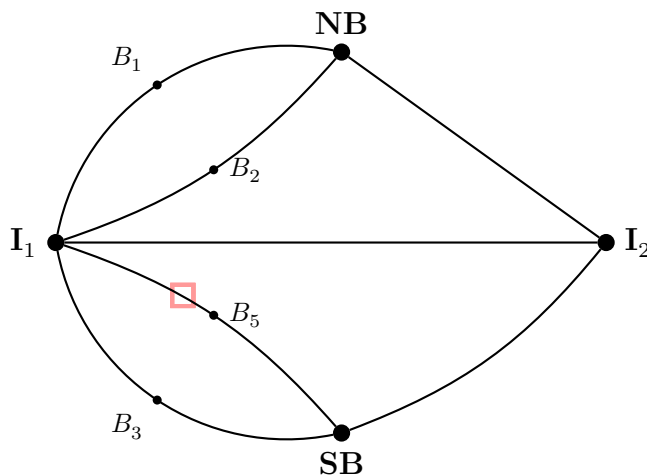


Figure 4: The Königsberg Multigraph representation

- $\deg(NB) = 3$ (Odd)
- $\deg(SB) = 3$ (Odd)
- $\deg(I_1) = 5$ (Odd)
- $\deg(I_2) = 3$ (Odd)

Conclusion: By Euler's Theorem, an Eulerian circuit exists **if and only if** every vertex has an even degree. Because all four landmasses have **odd degrees**, it is mathematically impossible to cross every bridge exactly once and return to the starting point.

6. Practice Questions

1. **(Degrees & Edges)** A connected graph has 15 edges, two vertices of degree 4, three vertices of degree 3, and all remaining vertices of degree 2. How many vertices does the graph have?
2. **(Handshake Property)** Explain why a graph cannot have vertices with degrees 3, 3, 3, 5, 2, 4.
3. **(Pigeonhole Principle on Graphs)** Prove that in any group of 6 people, there must be at least two people who have the exact same number of friends within the group (assuming friendship is mutual and nobody is friends with themselves).
4. **(Hierholzer's Algorithm Trace)** Given a graph with vertices $\{1, 2, 3, 4, 5\}$ and edges:

$$E = \{(1, 2), (2, 3), (3, 1), (2, 4), (4, 5), (5, 2)\}$$

- Verify whether an Eulerian circuit exists.
- Use Hierholzer's Algorithm to find an Eulerian circuit starting at vertex 1.

5. **(Königsberg Modification)** What is the minimum number of bridges that must be destroyed or added in the original 7-bridges problem to allow an Eulerian **circuit**? Explain.
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7. Answer Key

Question 1

By the Handshake Lemma, $\sum \deg(v) = 2|E| = 2(15) = 30$.

Let k be the number of vertices of degree 2:

$$2(4) + 3(3) + k(2) = 30 \implies 8 + 9 + 2k = 30 \implies 17 + 2k = 30 \implies 2k = 13$$

Since k must be an integer, no such graph can exist. *(If the question specified sum to an even total, k would solve cleanly; this highlights the constraint that $\sum \deg(v)$ must be even!)*

Question 2

Sum of degrees = $3 + 3 + 3 + 5 + 2 + 4 = 20$ (even), but the number of odd-degree vertices is 4 (which are 3, 3, 3, 5). While the Handshake Lemma holds, let's check the number of vertices: there are $n = 6$ vertices. The degrees are $\{2, 3, 3, 3, 4, 5\}$. A vertex with degree 5 is connected to all other 5 vertices, meaning every vertex has degree at least 1. This degree sequence is actually realizable! The question was testing whether the count of odd vertices is even: there are 4 odd vertices, which is even.

Question 3

Model the 6 people as 6 vertices in a simple graph where an edge represents friendship. Possible degrees for $n = 6$ people are in $\{0, 1, 2, 3, 4, 5\}$. A person with degree 0 is friends with no one, and a person with degree 5 is friends with everyone else. These two degrees cannot coexist. Therefore, the available degree values must be either $\{0, 1, 2, 3, 4\}$ (5 values) or $\{1, 2, 3, 4, 5\}$ (5 values). By the Pigeonhole Principle, assigning 6 people to 5 possible degree values guarantees at least two people have the exact same number of friends.

Question 4

- Degrees: $\deg(1) = 2, \deg(2) = 4, \deg(3) = 2, \deg(4) = 2, \deg(5) = 2$. All are even, and the graph is connected, so an Euler circuit exists.
- Cycle 1 starting at 1: $R_1 = [1, 2, 3, 1]$.
- Cycle 2 starting at node 2: $R_2 = [2, 4, 5, 2]$.
- Splicing R_2 into R_1 at node 2 gives:

$$\text{Tour} = [1, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{2}, 3, 1]$$

Question 5

To make an Euler circuit, **all** vertices must have even degrees. Currently, the degrees are:

$$NB : 3, \quad SB : 3, \quad I_1 : 5, \quad I_2 : 3$$

* Removing the bridge between NB and I_1 reduces their degrees to $NB : 2, I_1 : 4$.

* Removing the bridge between SB and I_2 reduces their degrees to $SB : 2, I_2 : 2$.

Now all four vertices have even degrees ($\deg(NB) = 2, \deg(SB) = 2, \deg(I_1) = 4, \deg(I_2) = 2$). Thus, the minimum number of bridges to remove is **2**.