CS 290 Market Design	**Fall 2020**

Lecture 1

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1.1 Overview of One-Sided Matching Markets

One-Sided Matching markets involve a set A of agents and set G of indivisible goods. Each agent defines a preference over the set of goods. For simplicity assume that $|A| = |G| = n$. We will consider a variety of different settings, outlined below, for which we want to find a mechanism, running in polynomial time, that takes in the preference lists and outputs a perfect matching. Additionally, the mechanism should satisfy certain "good" properties that we describe below.

One characteristic of different settings is whether agents specify their preferences in an ordinal or cardinal manner.

Definition 1.1. ***Ordinal Preferences:*** *Each agent $a_i \in A$ represents its preferences as a list, l, of goods such that $l(a_i) = \pi(G)$ where π is some permutation of goods in G.*

Definition 1.2. ***Cardinal Preferences:*** *Each agent $a_i \in A$ defines its utility over goods in G, such that $\forall a_i \in A, g_j \in G$, we have a non-negative utility $u_{ij} \geq 0$.*

Another way settings can differ is in whether the agents have an initial endowment or not. We provide an illustration of the different mechanisms for different market models and preference lists in the following table.

Initial Endowment / Preferences	No Endowment	Endowment
Ordinal	(a) Serial Dictatorship (b) Randomized S.D (c) Probabilistic S.D	TTC
Cardinal	Hylland-Zeckhauser	ϵ-approximate ADHZ

Definition 1.3. ***Dominant-strategy incentive compatible (DSIC):*.** *A DSIC mechanism is one such that regardless of the preferences reported by other agents, an agent can do no better than report its true preference list, i.e., truth-telling is a dominant strategy for all agents.*

Definition 1.4. ***Individual Rationality:*** *Each agent must weakly improve from initial allocation*

Definition 1.5. ***Pareto Optimality:*** *An allocation μ is pareto-optimal if $\nexists\ \mu'$ s.t. $\forall\ a_i,\ \mu'(a_i) \succeq_{a_i} \mu(a_i)$ and $\exists\ a_i$ s.t. $\mu'(a_i) \succ \mu(a_i)$*

Definition 1.6. ***Core-Stability:*** *An allocation μ is core-stable if $\forall S \subseteq A$, there does not exist a perfect matching*

μ_S on $(S, h(S))$, where $h(S)$ is the initial set of houses for agents in S, such that

$$\forall i \in S, \mu_S(i) \succeq_i \mu(i)$$
$$\exists i \in S, \mu_S(i) \succ_i \mu(i)$$

Remark 1.7. *Any allocation that has Core-Stability is also Pareto-Optimal (Consider the case when $S = A$)*

As an example of a market with initial endowment and ordinal preferences we study the Housing Problem.

Problem 1.8. ***(Housing Problem)*** *We are given a set A of agents, and set H of houses and an initial allocation μ_0. Each agent a_i has a totally ordered preference list $\succ_i$ over all houses in H. We wish to design a mechanism that weakly improves the allocation of all agents.*

Theorem 1.9. *There exists a mechanism $\mathcal{M}$ for the housing problem such that:*

(a) *$\mathcal{M}$ runs in polynomial time*

(b) *$\mathcal{M}$ is individually rational*

(c) *$\mathcal{M}$ is DSIC*

(d) *$\mathcal{M}$ provides an allocation with core stability (and thus Pareto-Optimal)*

1.2 Top Trading Cycle (TTC)

We present the TTC algorithm as a mechanism for the Housing Problem. Every agent draws a directed edge to the agent who owns the house it desires the most. If an agent prefers its house the most it draws an edge to itself. The resulting graph contains at least one cycle, and a cycle can be found in linear time. Pick an arbitrary cycle and for each agent in the cycle assign it the house of the agent it points to. After removing these agents and their houses from the market, recurse on the remaining agents and houses.

Top Trading Cycle(TTC):
Input: $(A, G, \succ)$, initial allocation μ_0
Output: Perfect Matching μ which is individually rational, DSIC and Core-Stable for all agents.

1. L = A
2. While $L \neq \emptyset$:
   (a) $E = \emptyset$.
   (b) $\forall\ a_i \in L,\ E = E \cup (a_i \to a_j)$, where a_j owns house most-preferred by a_i in $\mu_0(L)$).
   (c) Find a cycle, C, in $G = (A, E)$.
   (d) $\forall\ (a_i \to a_j) \in C,\ \mu(a_i) = \mu_0(a_j)$.
   (e) Remove all a_i from L.
3. Return μ.

Figure 1.2: Gale's Mechanism for Housing Problem

Lemma 1.10. *A directed graph with each vertex having one outgoing edge contains a cycle*

Proof. Start at some initial vertex and follow its outgoing edge to new vertex. Since each vertex has an outgoing edge you can keep traversing on this path till you return to an already visited vertex. Such a vertex must exist since after $n - 1$ edges you have visited all vertices. This completes the cycle □

Lemma 1.11. *TTC terminates in polynomial time*

Proof. By construction, at every iteration of the graph each node has exactly one out-going edge. From Lemma 1.10 such a graph will always have at least one cycle. As a result at every iteration we can remove a cycle and the algorithm terminates in at most n iterations. Finding a cycle in each iteration can be done in linear time.

□

Lemma 1.12. *TTC is individually rational*

Proof. Any agent a_i never adds an edge to a house it prefers less than its initial allocation. Thus in the iteration where a cycle containing a_i is resolved, a_i must weakly improve. □

Lemma 1.13. *TTC is DSIC*

Proof. Let a_i be an agent who falsifies his preference list. Assume that under honest reporting, $\mu(a_i) = h_j$. The only sensible manipulation for a_i to make is choosing a different permutation ordering of houses he prefers over h_j. Let a_i choose one such permutation, π, such that he now gets matched to a house,

h_k, which he prefers over h_j. We show that a_i would then be matched to h_k or better under his honest preference list.

Consider the iteration under the falsified list where a_i forms a cycle with the owner of h_k and some other agents. Even if a_i did not complete the cycle all the other agents in this cycle would continue pointing to their neighbors in this cycle until a_i gets allocated. This is because the agent pointing to a_i will continue to do so until a_i's house is allocated, which will not occur until a_i receives h_k. Since h_k is ranked higher in π than in original preference list, under true reporting this cycle will still be available if a_i has not already been allocated a better house. Hence a_i must be allocated h_k or better under honest reporting and does not benefit by misreporting his preferences.

□

Lemma 1.14. *TTC provides Core-Stability*

Proof. Let μ be the matching obtained by TTC. Assume for contradiction that there exists a subset $S \subseteq A$ for which a better matching μ' exists for $(S, \mu(S))$ in the sense that $\forall i \in S, \mu'(i) \succeq_i \mu(i)$ and $\exists i, \mu'(i) \succ_i \mu(i)$.

Consider the first cycle removed by the algorithm that involves a vertex of S. Assume that not every vertex in this cycle lies within S; then there must be some vertex u pointing to a vertex v outside of S. But then this implies that u prefers the item held by v more than all of the items held within S, so u will prefer the resulting assignment μ to any assignment μ' on $(S, \mu(S))$. So we can consider cycles completely contained within S. But then every agent within S will be pointing to their most preferred available item within S at all times and hence we cannot have such a μ'. Hence TTC satisfies core stability. □