Online Orthogonal Vectors Revisited

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Columbia Theory Lunch, 2026

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Outline

- Online Orthogonal Vectors
- Algorithms
- Lower Bounds

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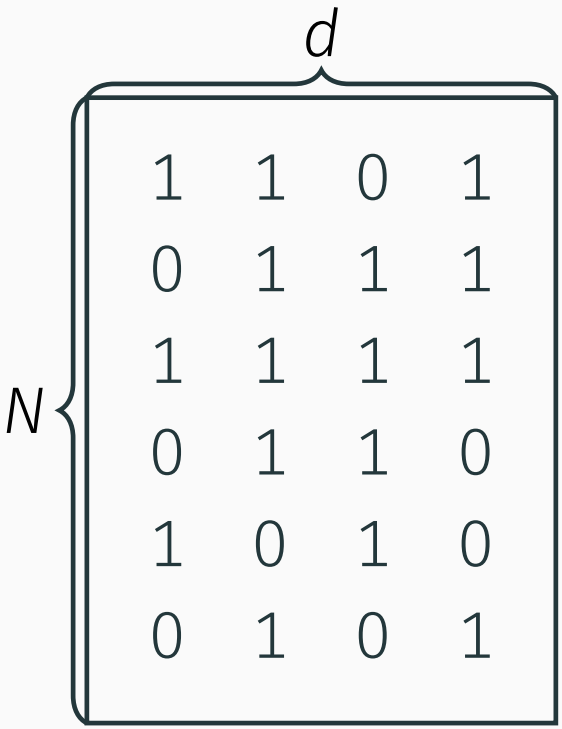
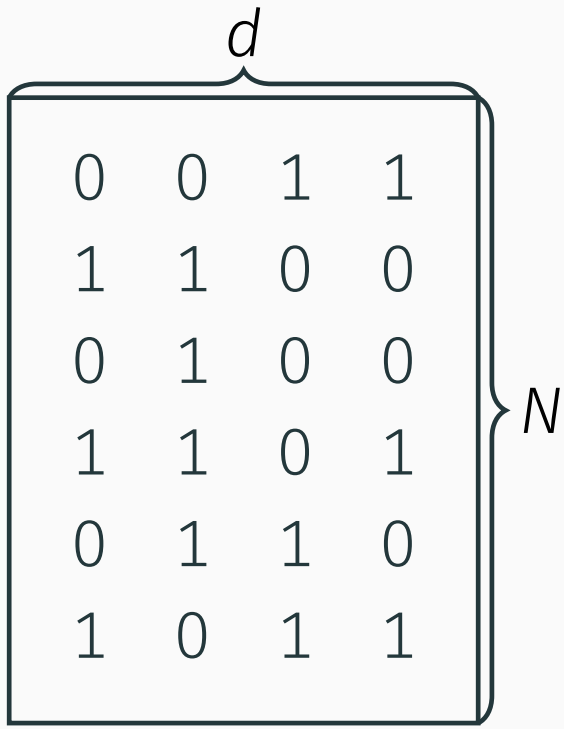
(Offline) Orthogonal Vectors

Orthogonal Vectors Problem





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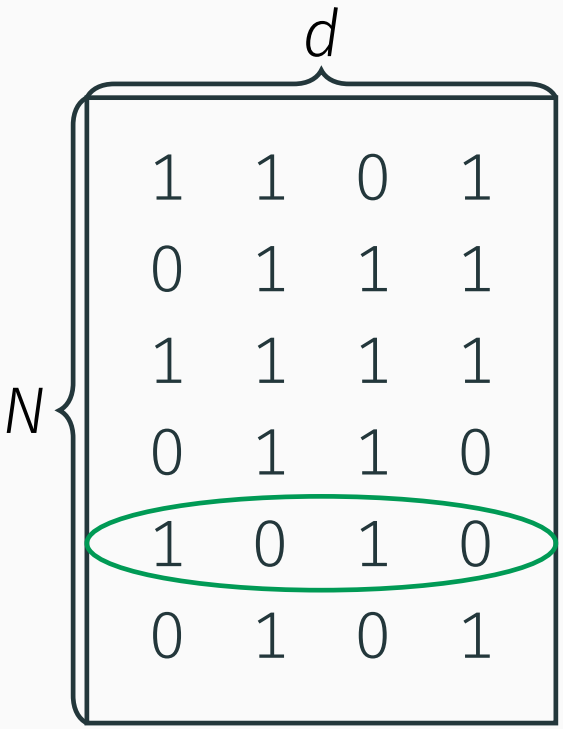

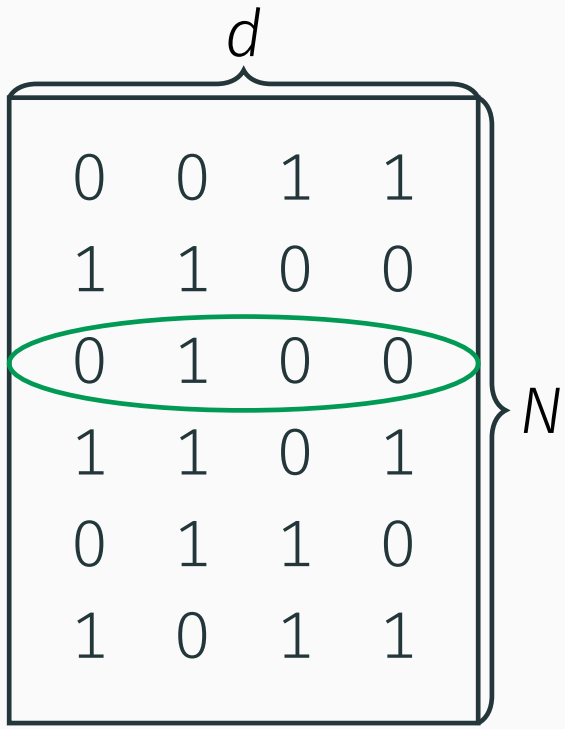

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- S, T are sets of N vectors from $\{0,1\}^d$. Are there $s \in S$ and $t \in T$ such that $s \cdot t = \sum_{i=1}^{d} s_i \cdot t_i = 0$?

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Is there a $N^{1.9999999}$ algorithm for OV?

Unconditional Lower bounds

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- Complexity Theory Dream: Prove Unconditional Lower bounds

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- We don’t really have a good toolkit yet to argue about lower bounds.

Boolean Satisfiability (SAT)

SAT

- $\varphi(x_1, \ldots, x_n) = (x_1 \vee \overline{x_2} \vee \ldots \vee x_7) \wedge$
$\ldots \wedge$
$(x_2 \vee \overline{x_3} \vee \ldots \vee x_8)$

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- n Boolean vars, m clauses
- $m = O(n)$
- k-SAT is SAT where clause length $\leq k$

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Corollary:

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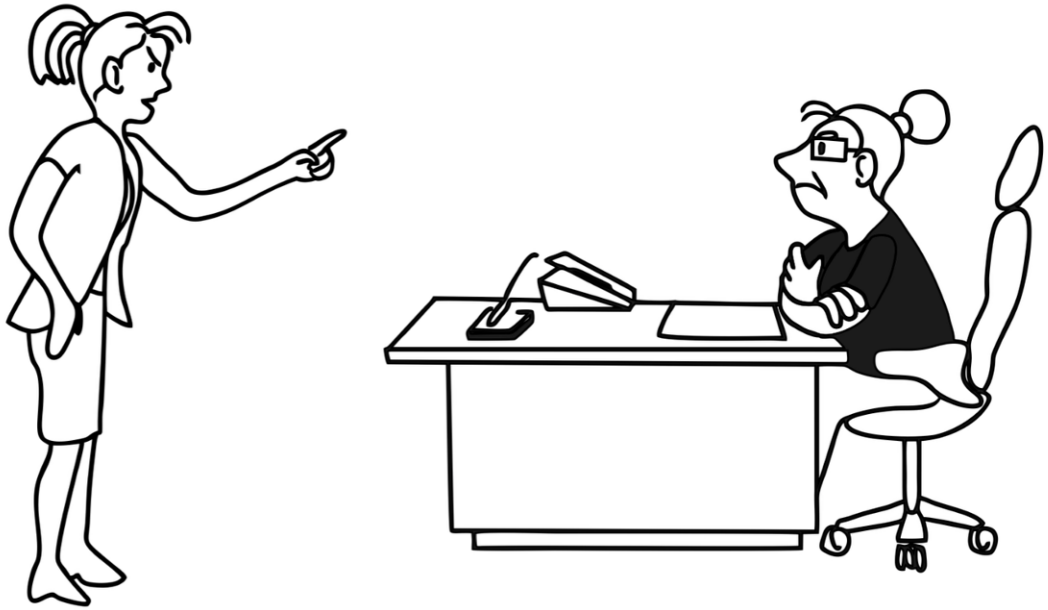
- YFP $=$ { Hamiltonian Path, Traveling Salesman, Independent Set, Clique, 3-Coloring, . . .}

Proof by Human Effort

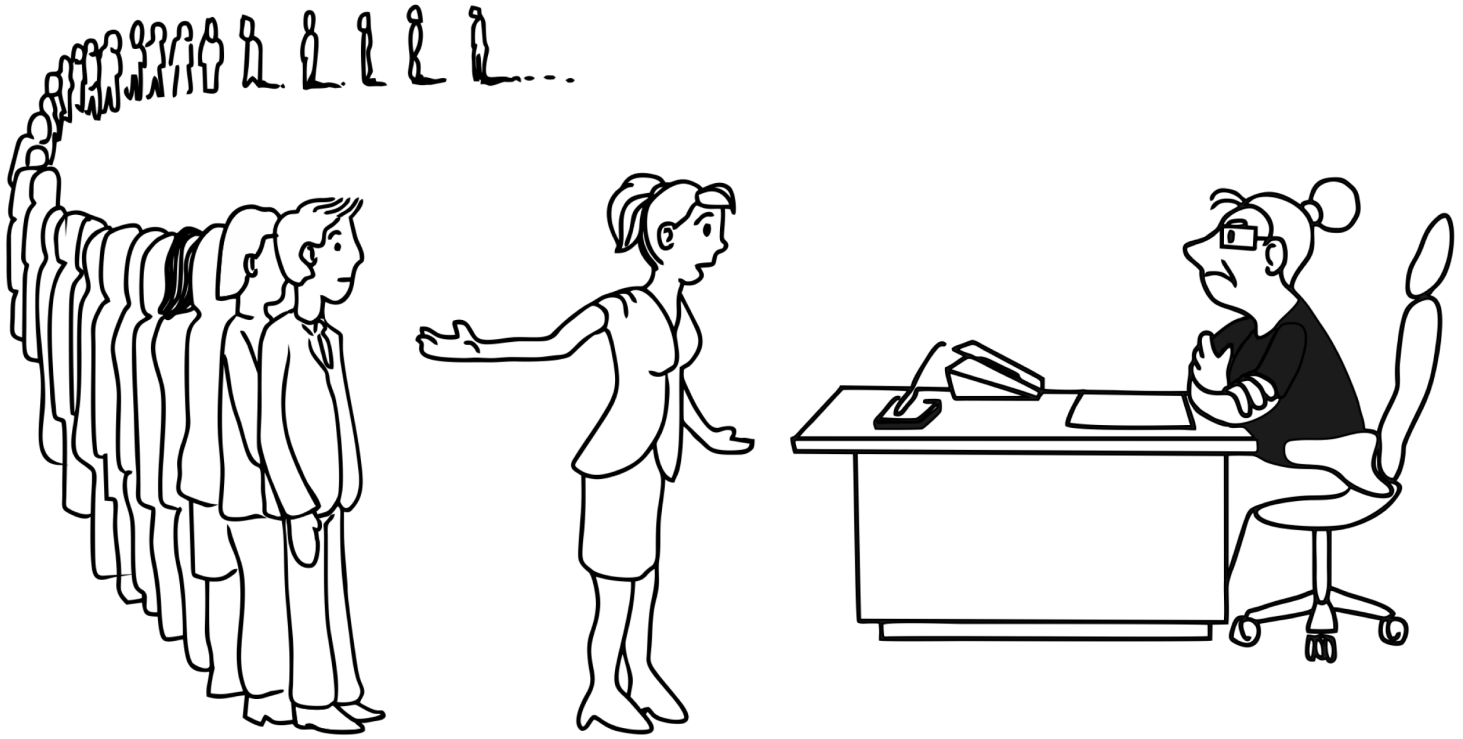
“I can’t find an efficient algorithm, I guess I’m just too dumb.”

Proof by Human Effort



“I can’t find an efficient algorithm, because no such algorithm is possible!”

Proof by Human Effort



“I can’t find an efficient algorithm, but neither can all these famous people.”

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More conditional lower bounds?

- SAT can be solved in time $O(2^n \cdot \mathbf{poly}(|\varphi|))$
- We don't know how to solve SAT exponentially faster: in time $2^{0.9999n}$ even when $m = O(n)$
- Can we prove conditional lower bounds on polynomial time problems?

Strong Exponential Time Hypotheses

Strong Exponential Time Hypothesis (SETH)

For every $\varepsilon > 0$, there exists k such that no algorithm solves k-SAT on formulas with n variables in time $2^{(1-\varepsilon)n}$

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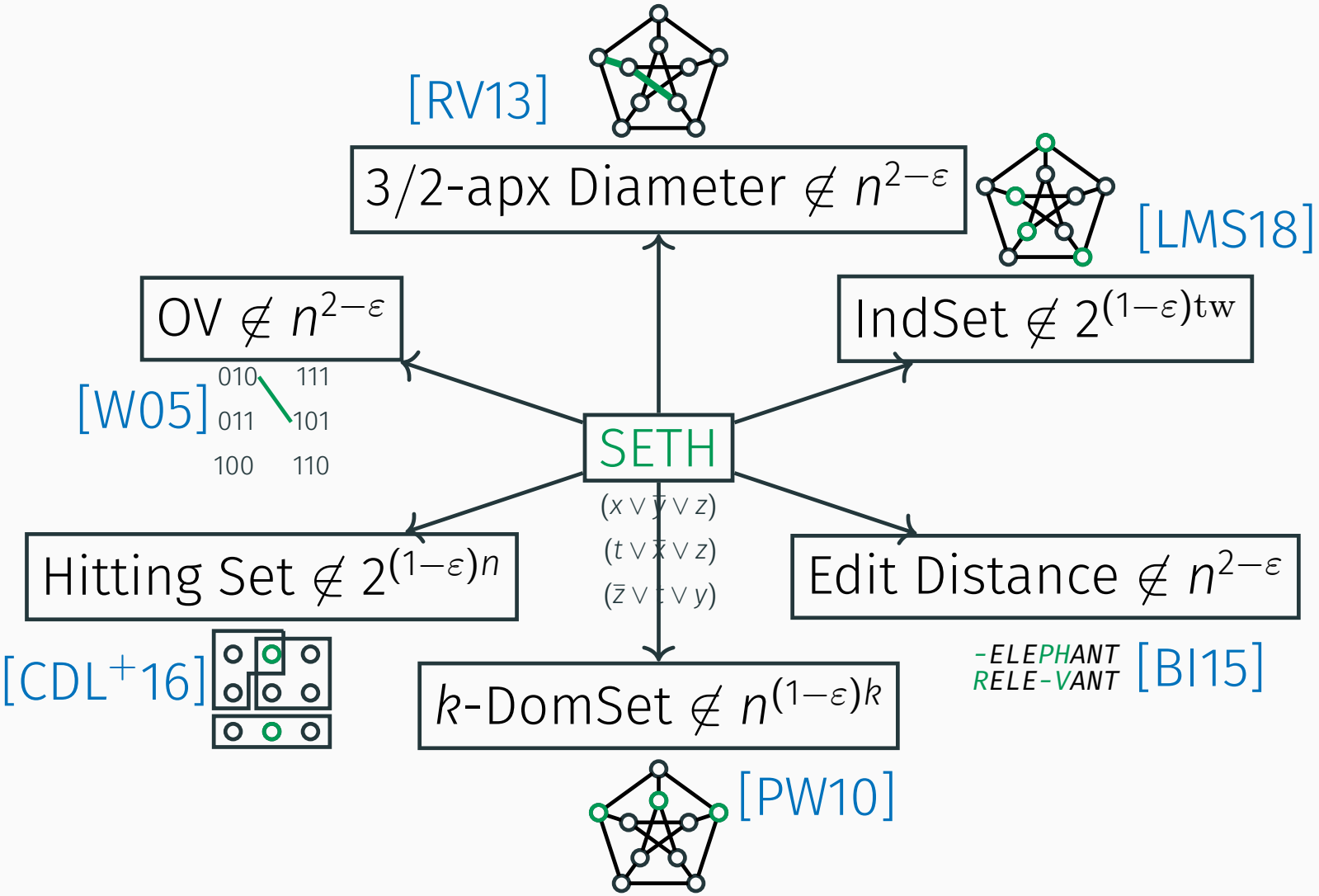
- SETH as a limit of current algorithmic techniques
- Refuting SETH implies super-linear circuit lower bounds

Fine-Grained Reductions



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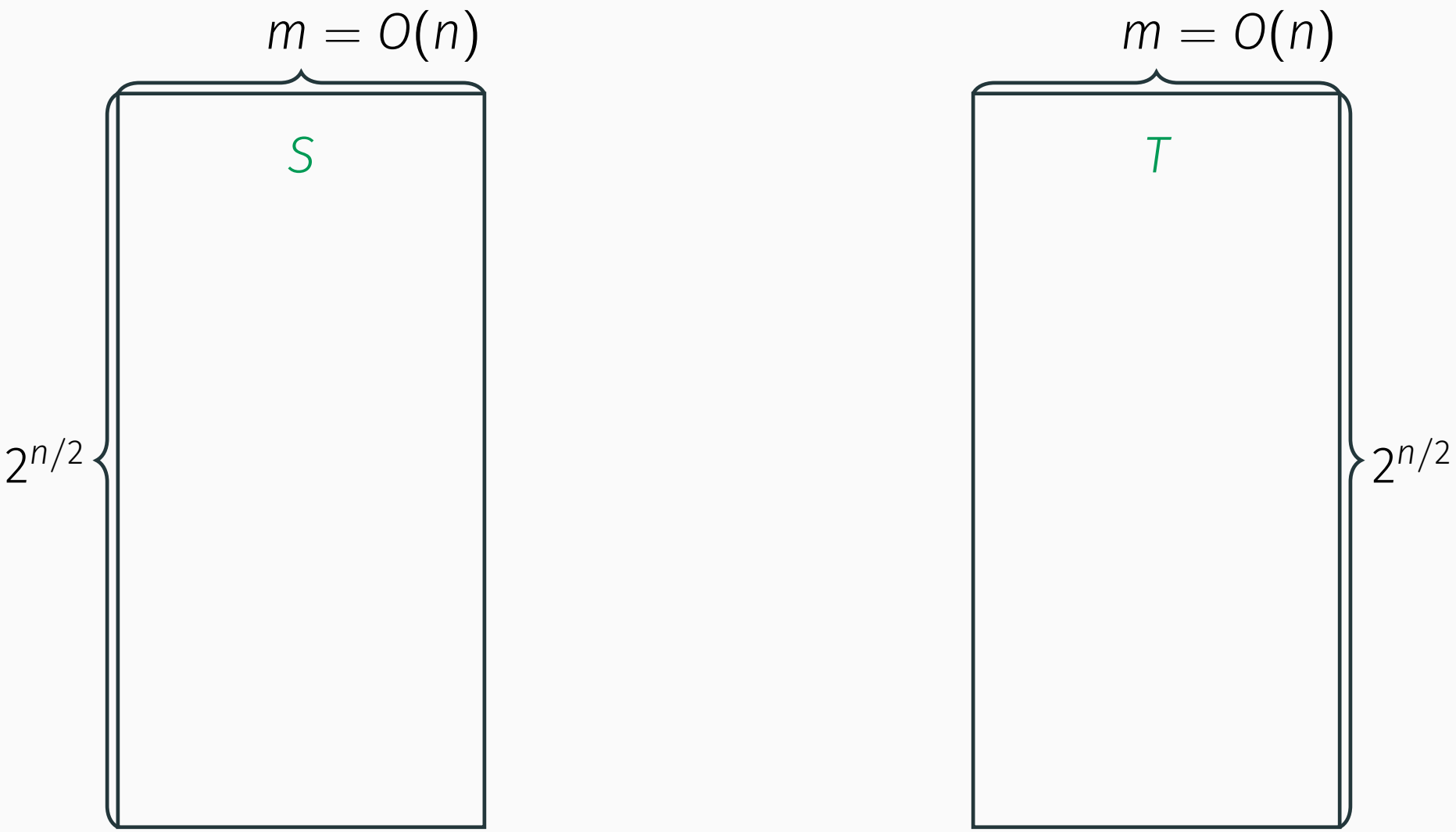

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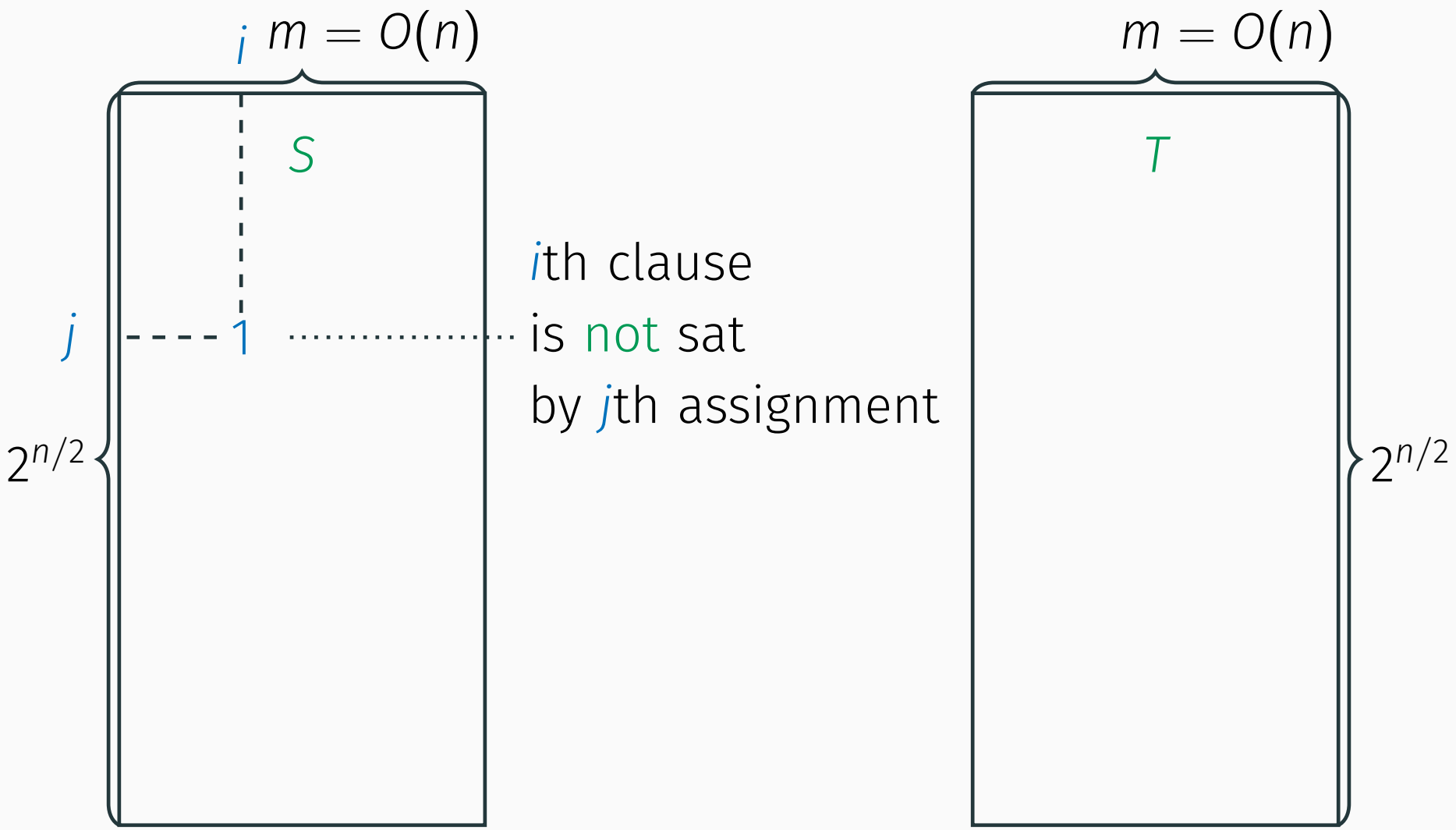

SETH $\Longrightarrow$ OV. Example

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_4) \wedge (\overline{x_4} \vee \overline{x_5} \vee \overline{x_6}) \wedge (x_1 \vee x_5 \vee x_6)$$

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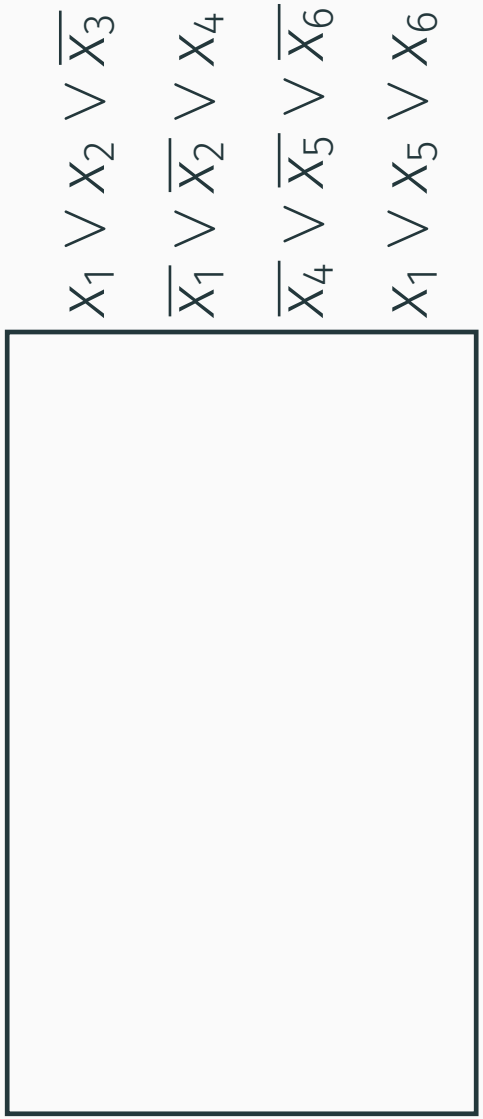

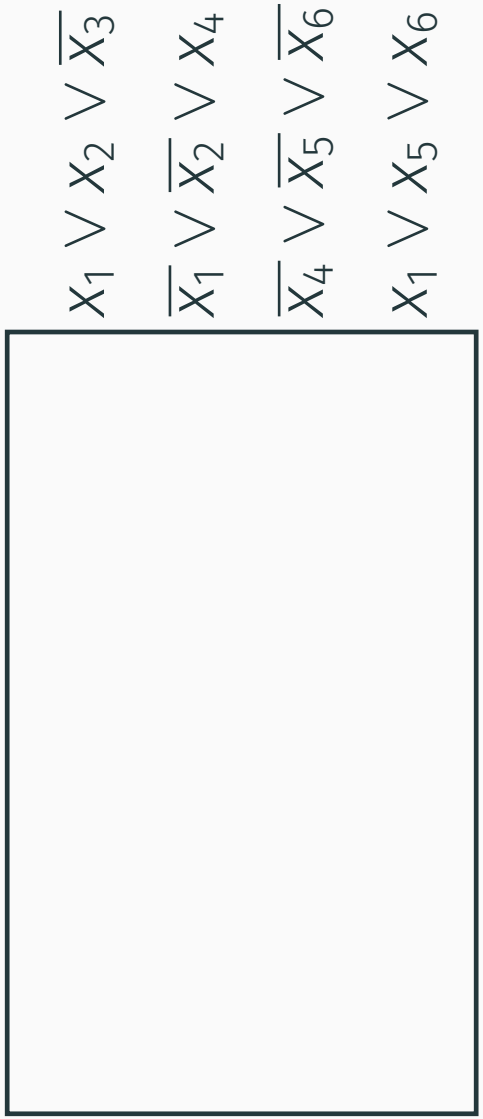

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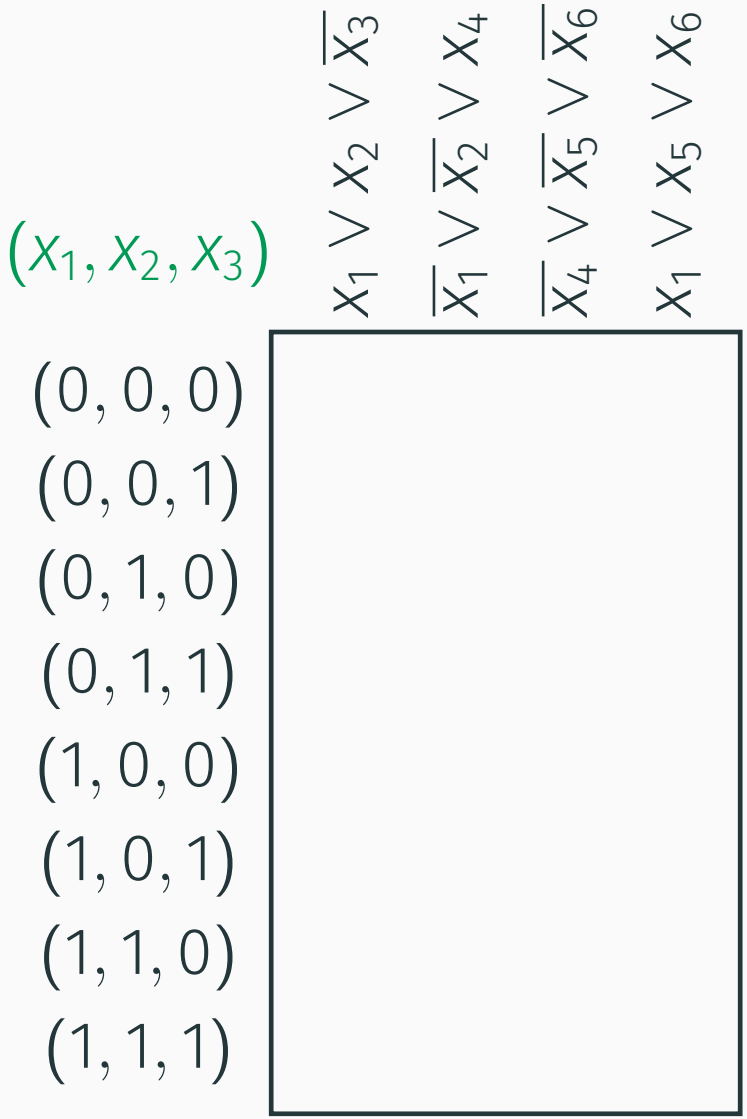

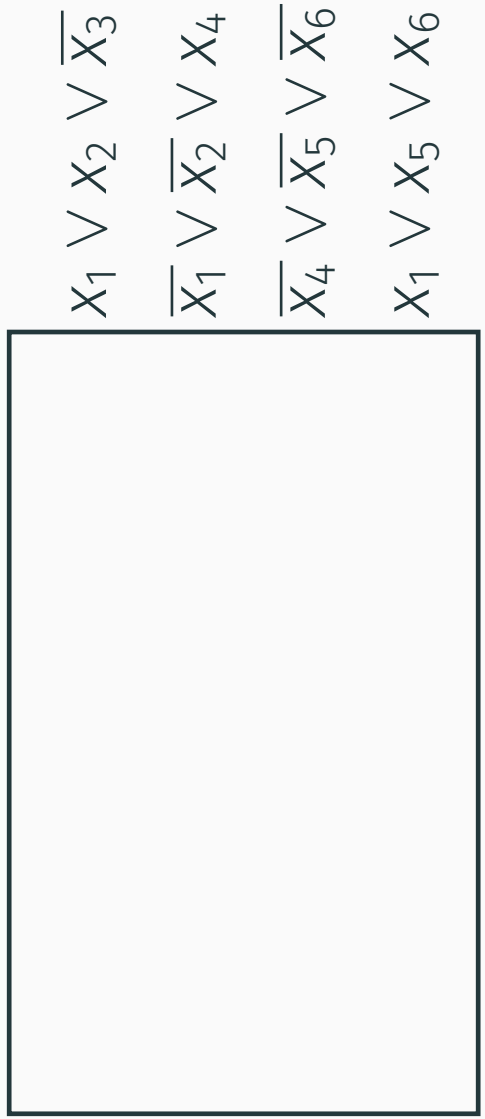

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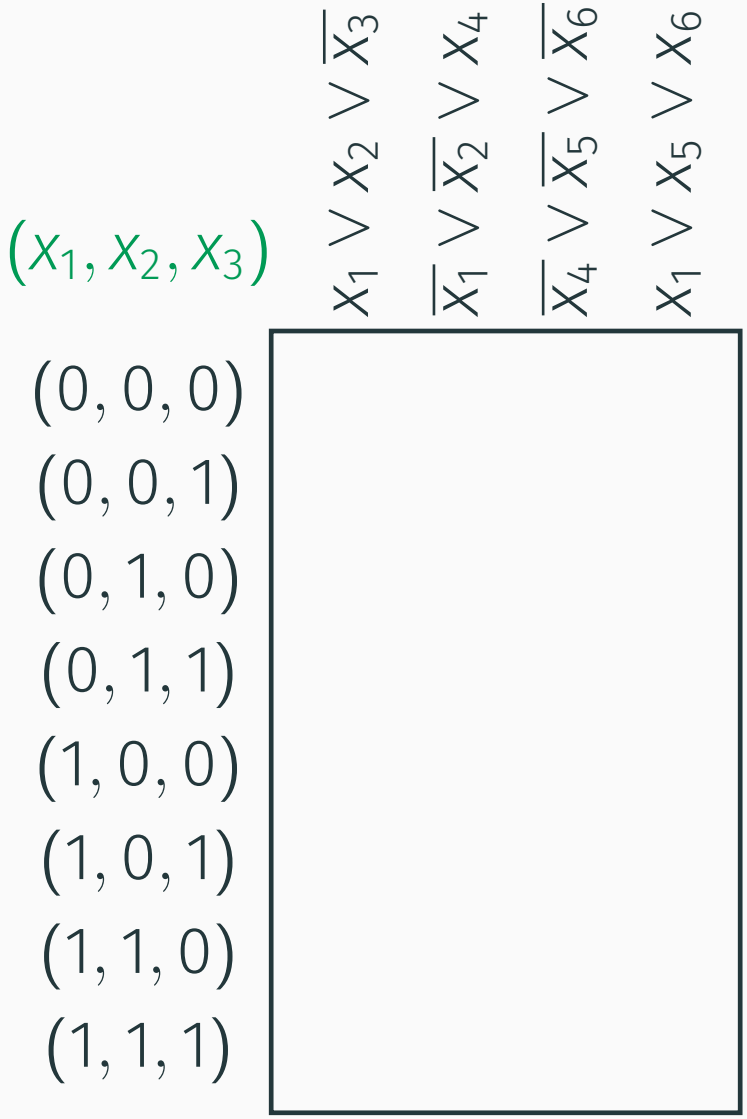

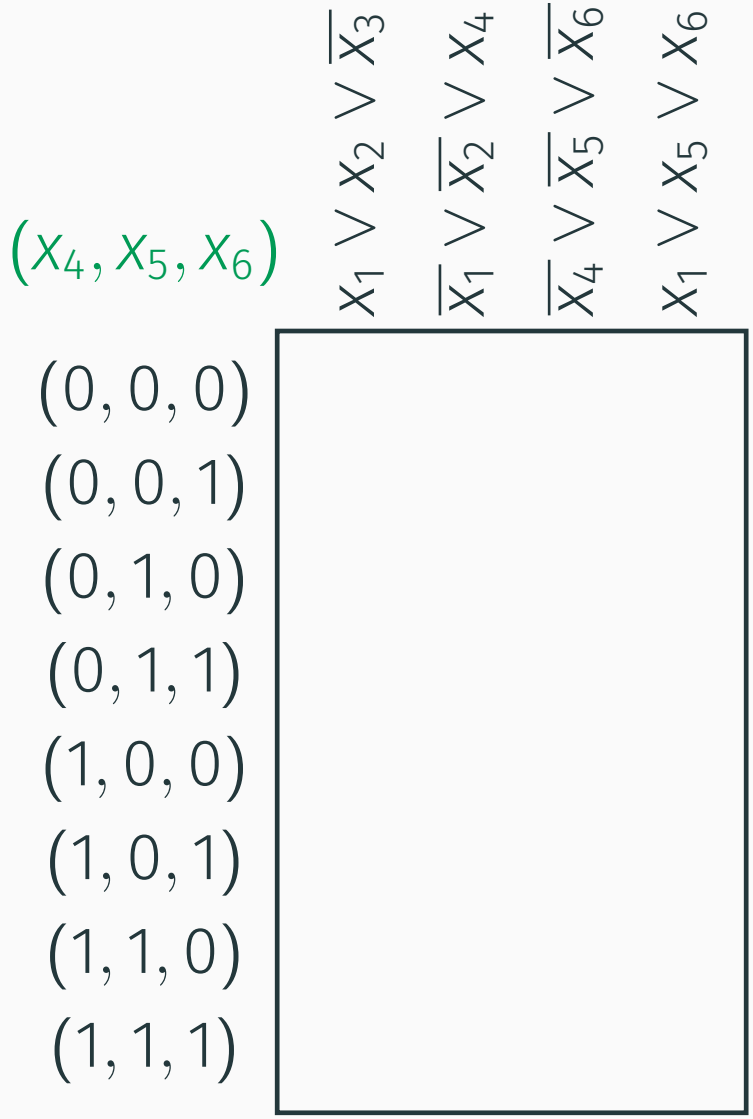

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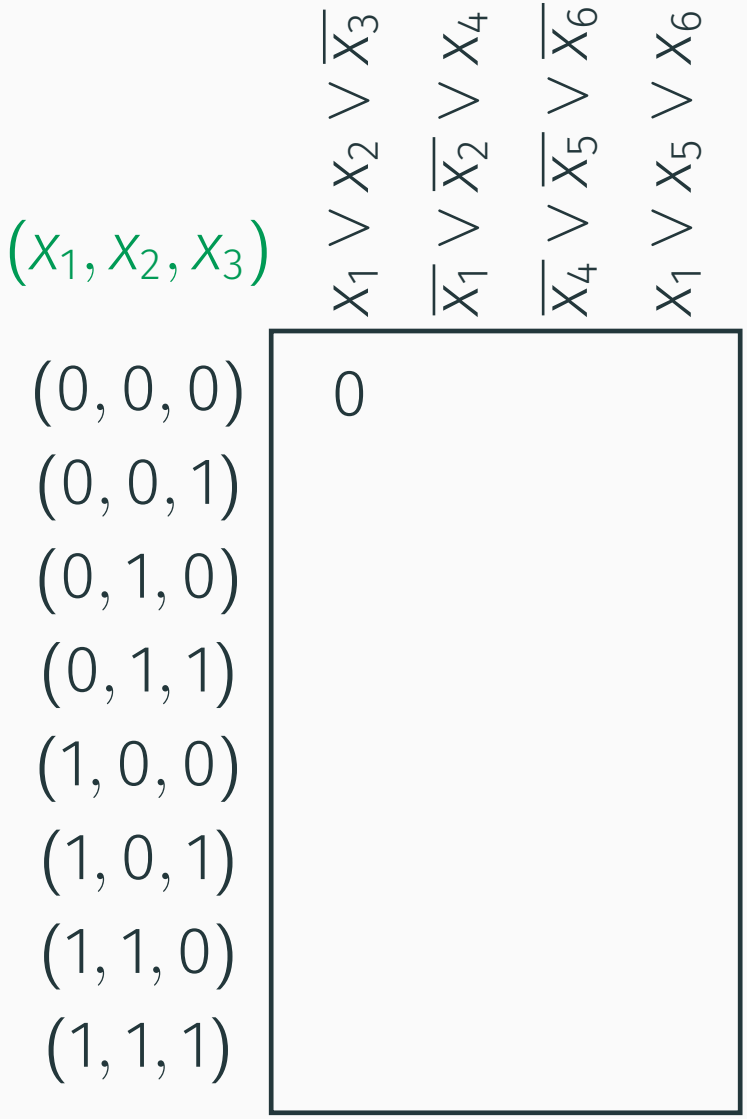

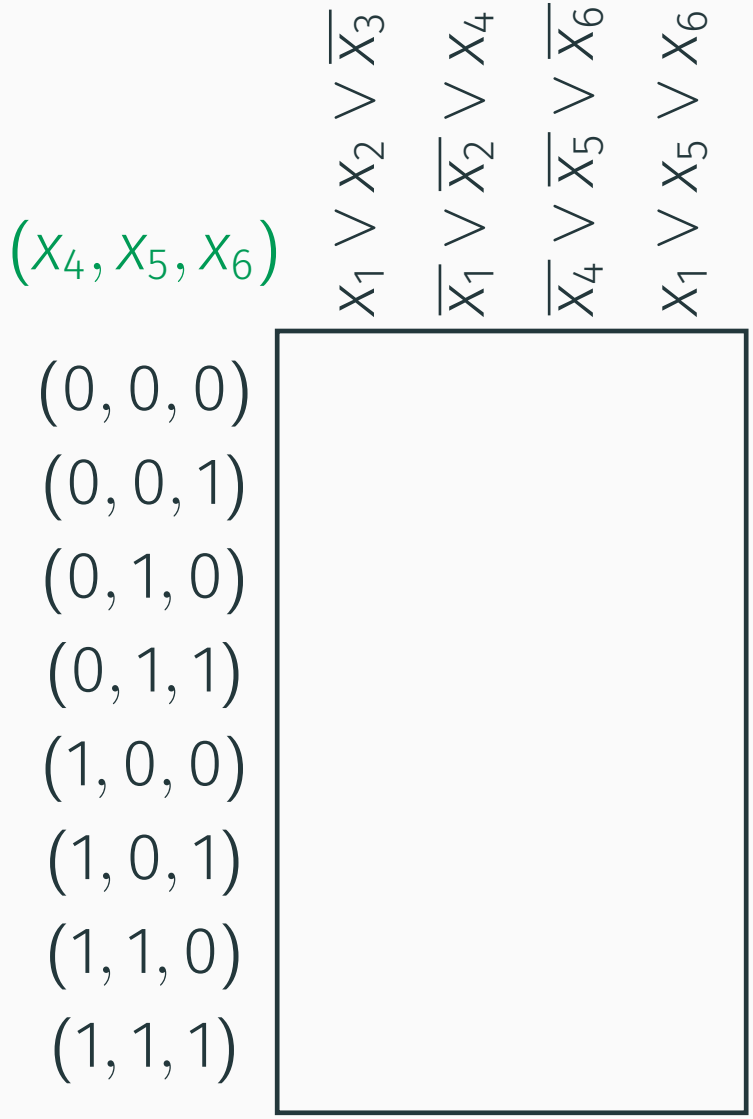

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(x_1, x_2, x_3)	$x_1 \vee x_2 \vee \overline{x_3}$	$\overline{x_1} \vee \overline{x_2} \vee x_4$	$\overline{x_4} \vee \overline{x_5} \vee \overline{x_6}$	$x_1 \vee x_5 \vee x_6$
$(0,0,0)$	0			
$(0,0,1)$	1			
$(0,1,0)$				
$(0,1,1)$				
$(1,0,0)$				
$(1,0,1)$				
$(1,1,0)$				
$(1,1,1)$				

(x_4, x_5, x_6)	$x_1 \vee x_2 \vee \overline{x_3}$	$\overline{x_1} \vee \overline{x_2} \vee x_4$	$\overline{x_4} \vee \overline{x_5} \vee \overline{x_6}$	$x_1 \vee x_5 \vee x_6$
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$(0,0,0)$	0	0	1	1
$(0,0,1)$	1	0	1	1
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SETH $\implies$ OV

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$$s_i = 1 \text{ iff } x \text{ doesn't satisfy clause } C_i$$

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$$N + N^{2-\varepsilon}$$

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$$N + N^{2-\varepsilon} = (2^{n/2})^{2-\varepsilon} = 2^{n(1-\varepsilon/2)}$$

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Orthogonal Vectors Conjecture

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- P vs NP is too hard
- Is SETH False? Beating brute force for SAT doesn't sound that crazy?
- Best algorithm for OV run in time $O(n^{2-1/\log c})$ [AWY14]

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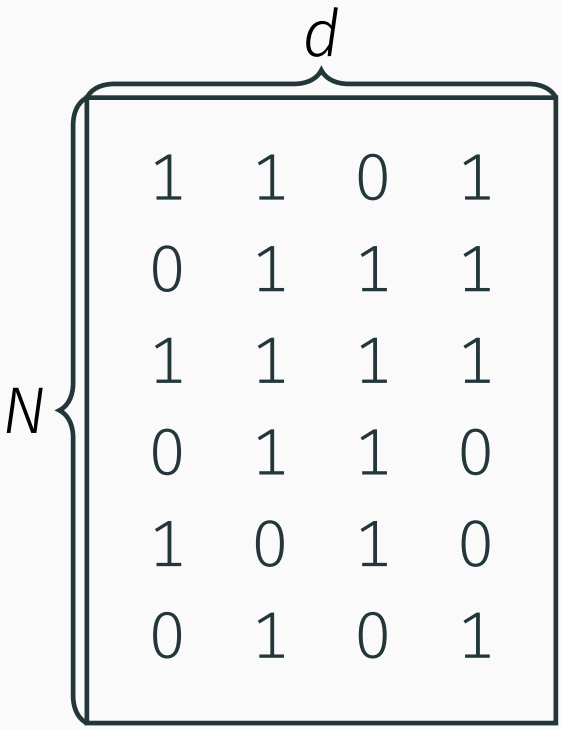

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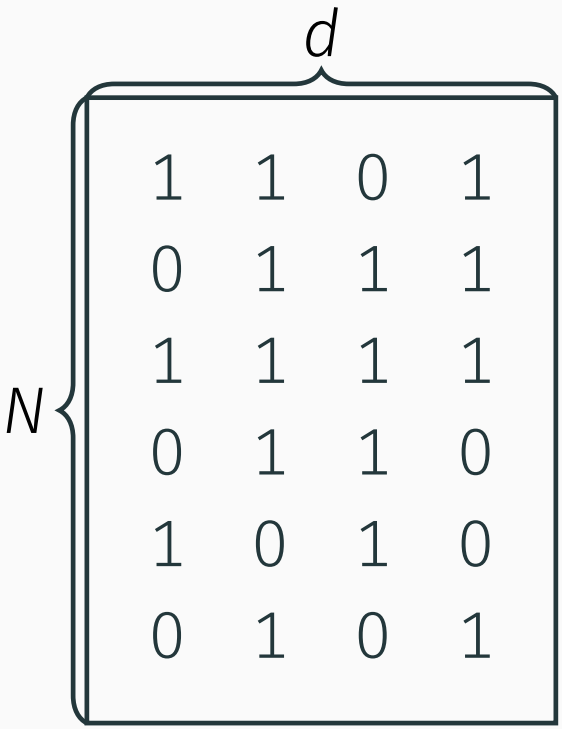

$q = 0100$

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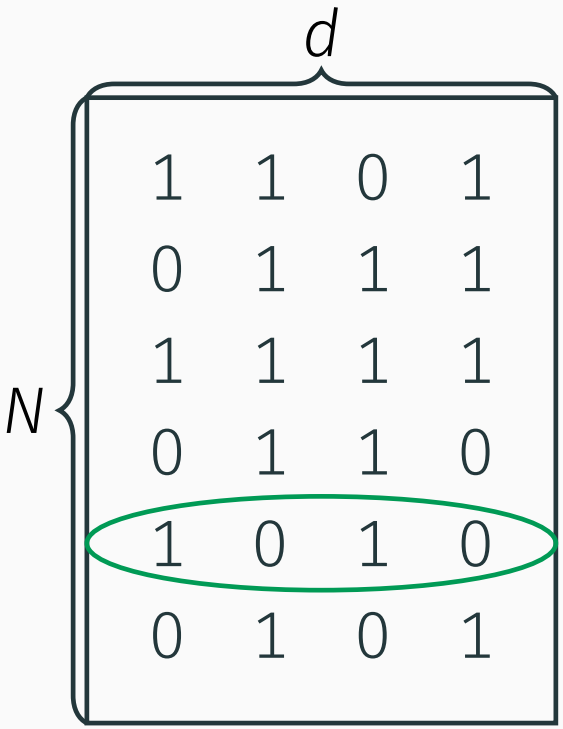

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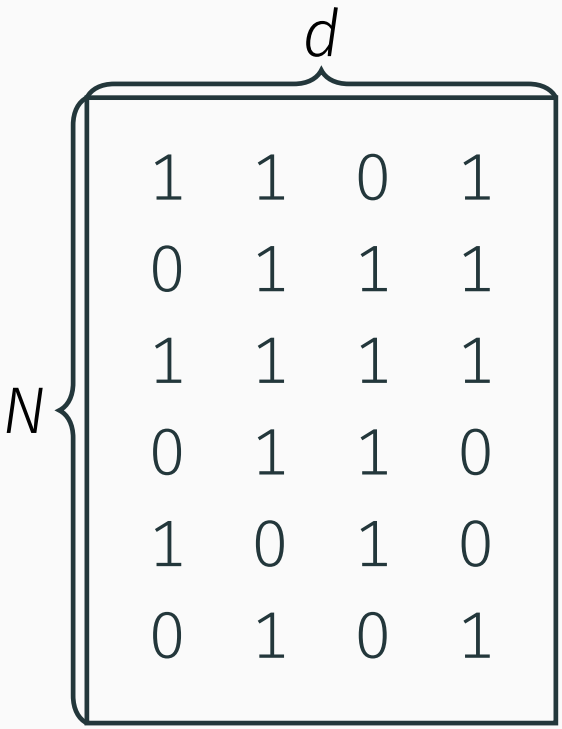

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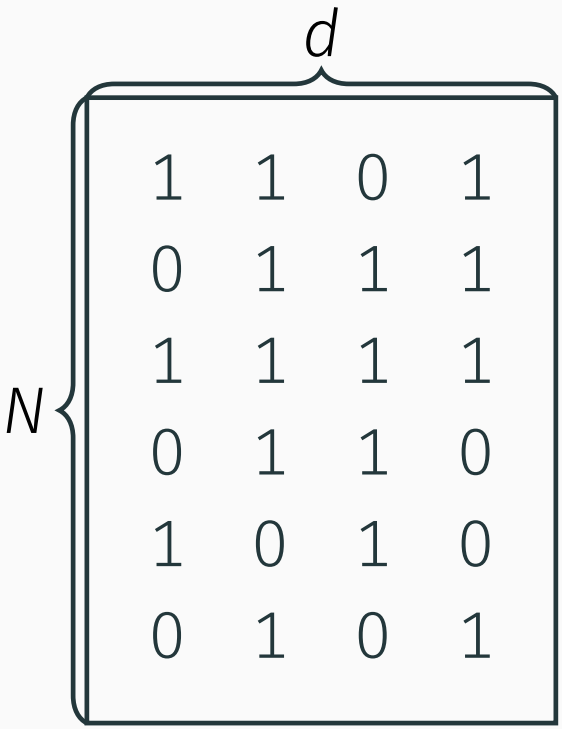

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No orthogonal vector

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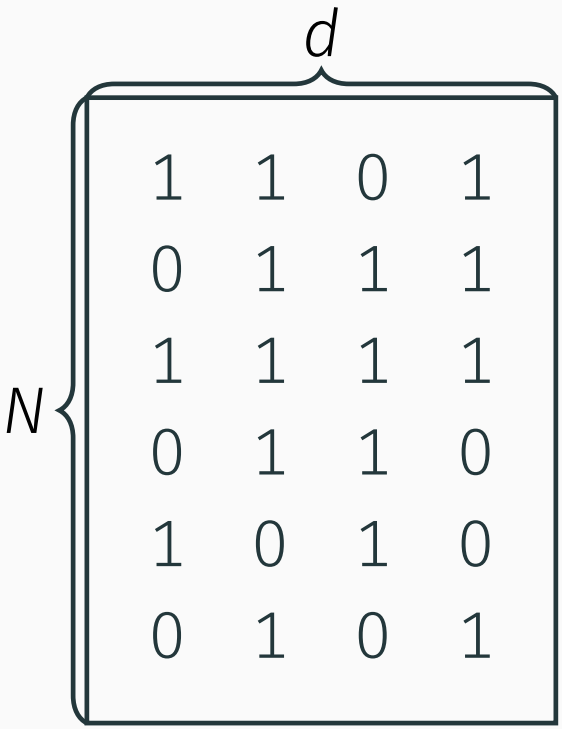
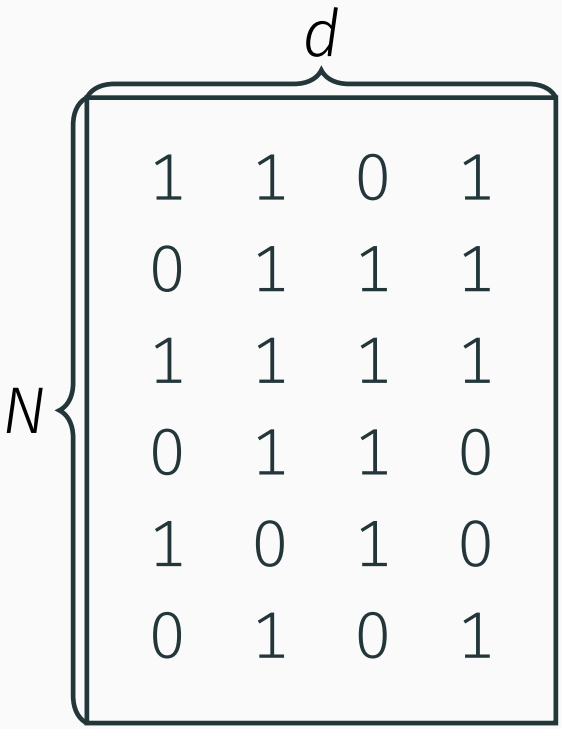


q

Online Orthogonal Vectors Problem



q

Does there exist input vector whose set of 0s contains q's 1s?

Equivalent Formulations

- SubsetQuery: preprocess n sets $S_1, \ldots, S_n \subseteq [d]$. For a query $q \subseteq [d]$, check if $\exists i : q \subseteq S_i$.

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- ContainmentQuery: preprocess n sets $S_1, \ldots, S_n \subseteq [d]$. For a query $q \subseteq [d]$, check if $\exists i : S_i \subseteq q$.

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- OnlineInnerProduct, OrthogonalRangeSearch, ApproximateNearestNeighbors, ...

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Outline

- Online Orthogonal Vectors
- Algorithms
- Lower Bounds

Average-case OnlineOV: Toy Algorithm

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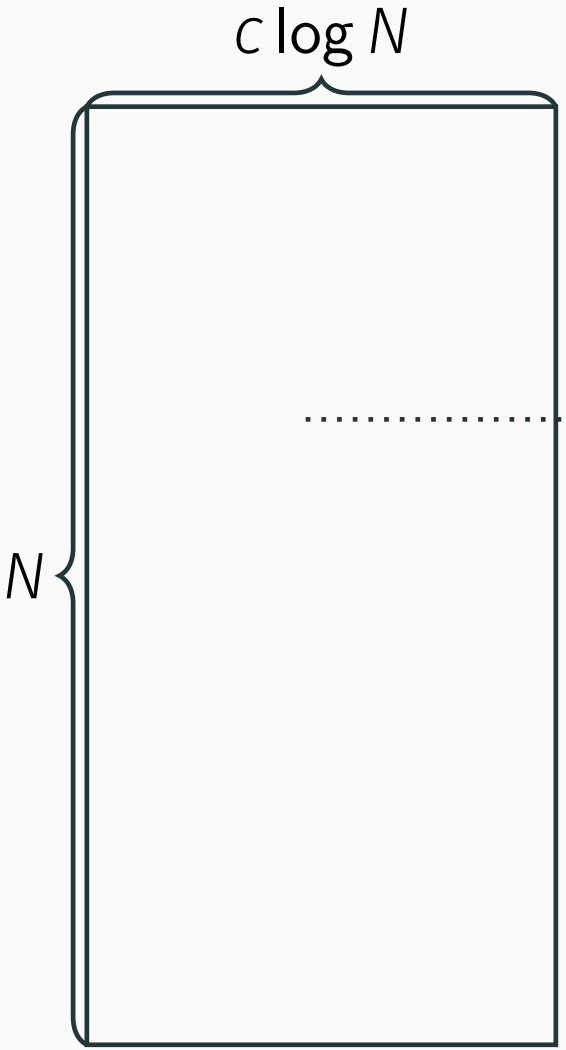

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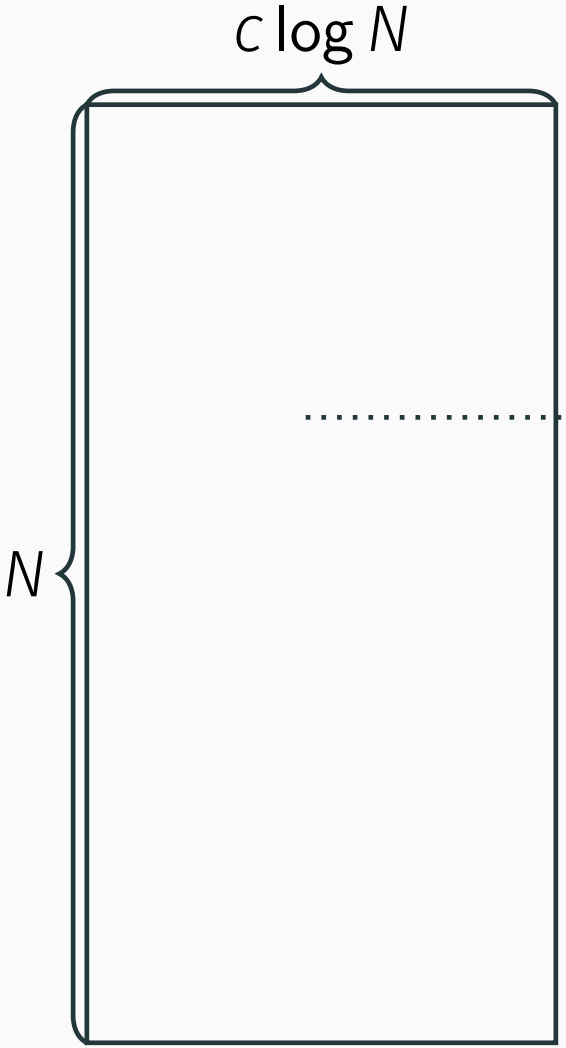

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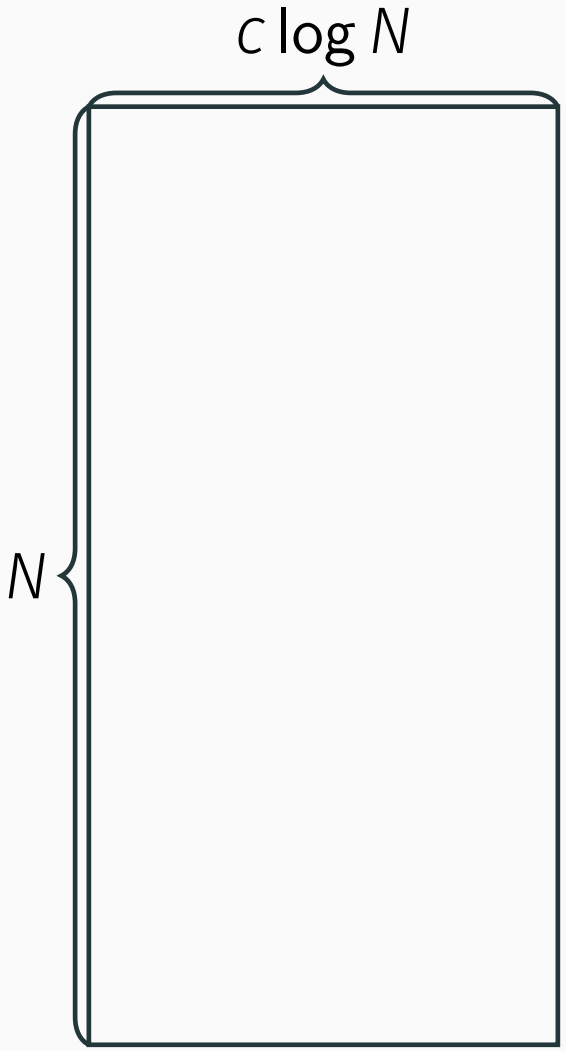

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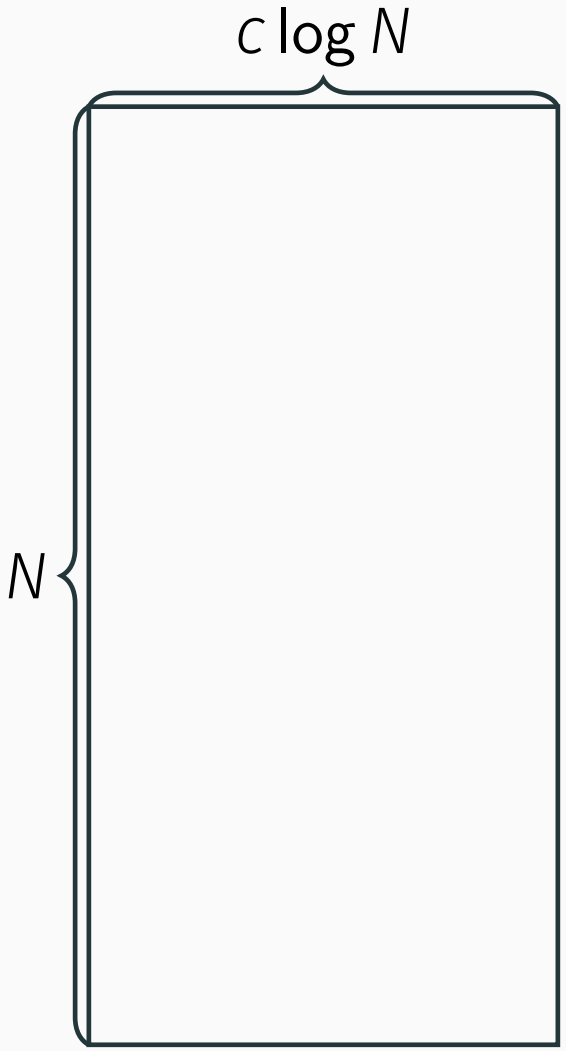

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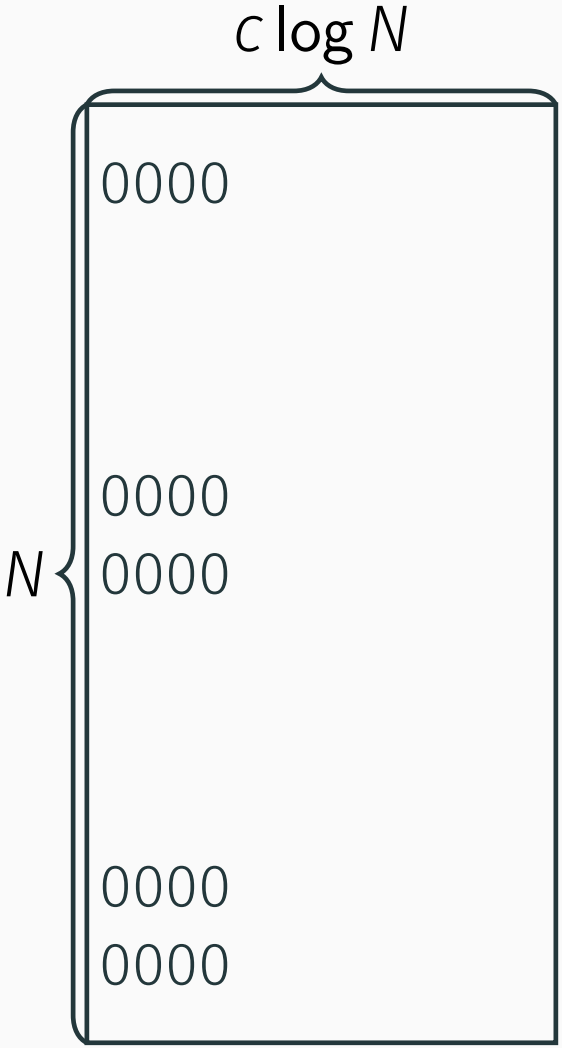

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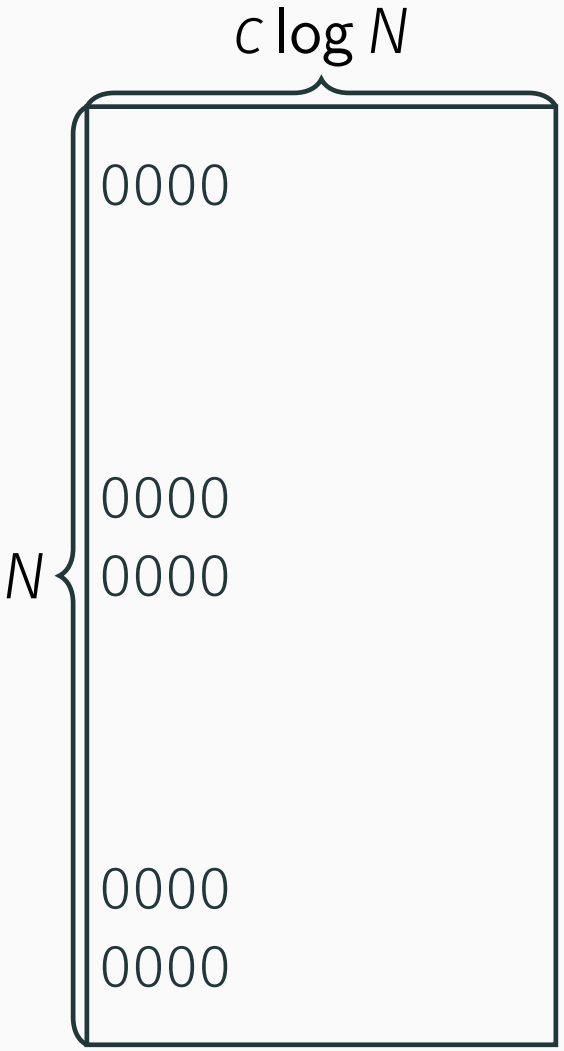

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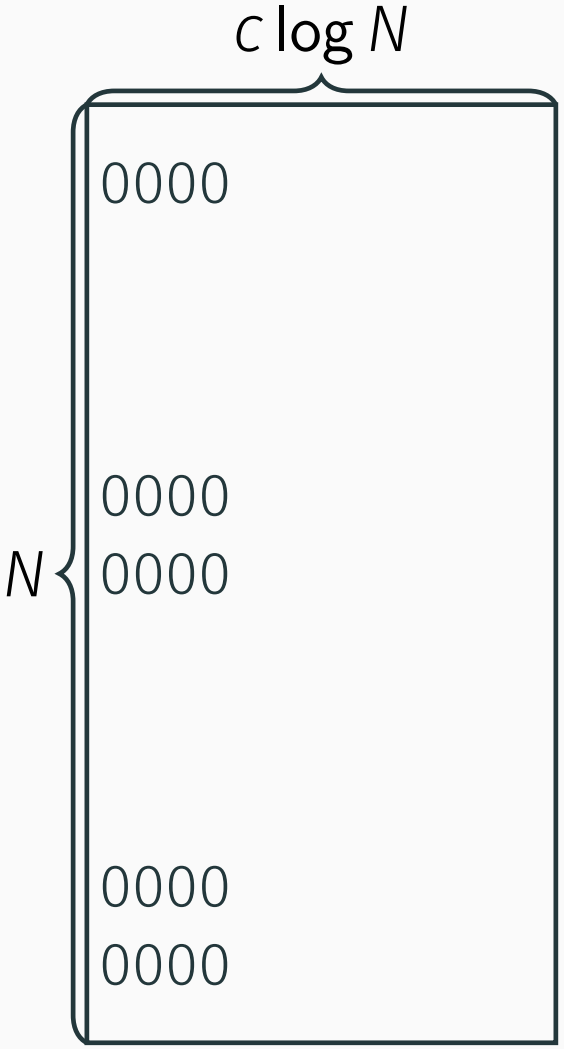

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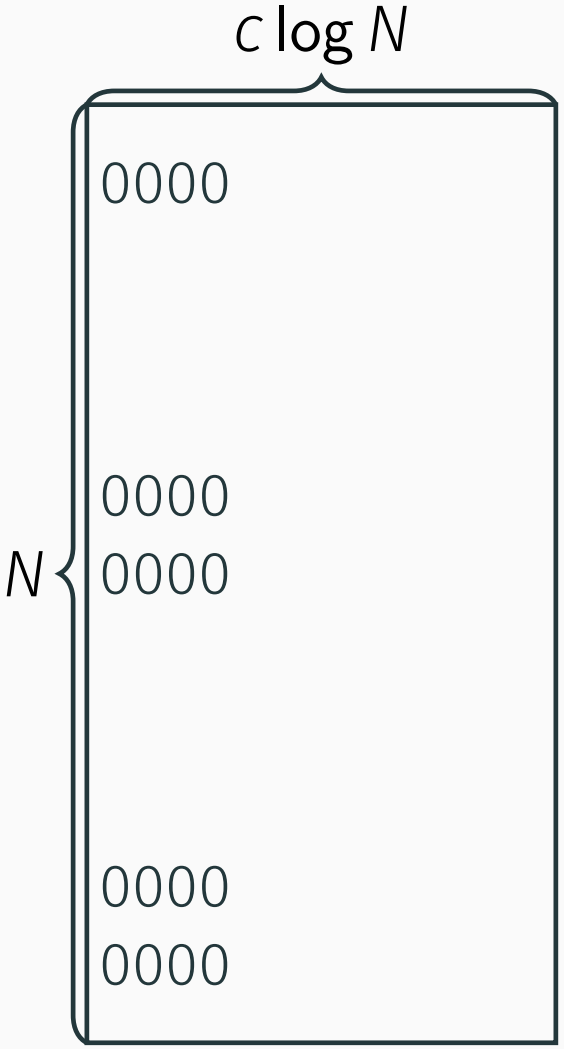

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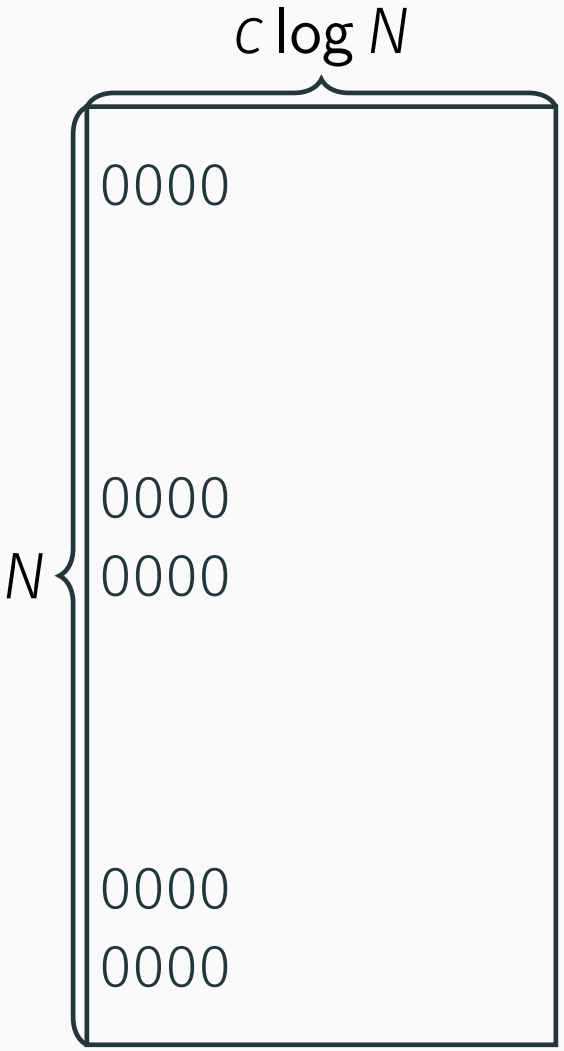

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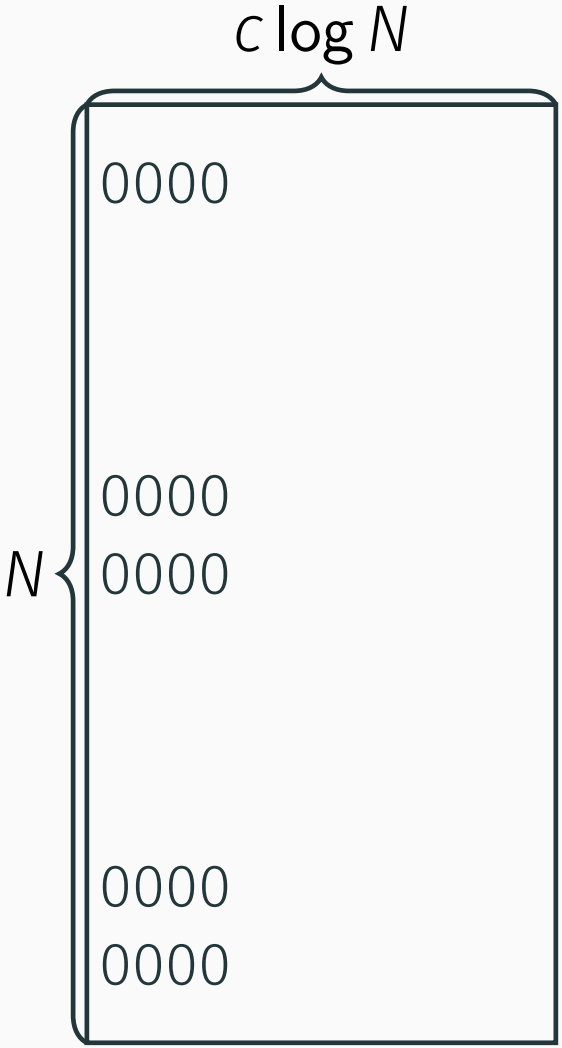

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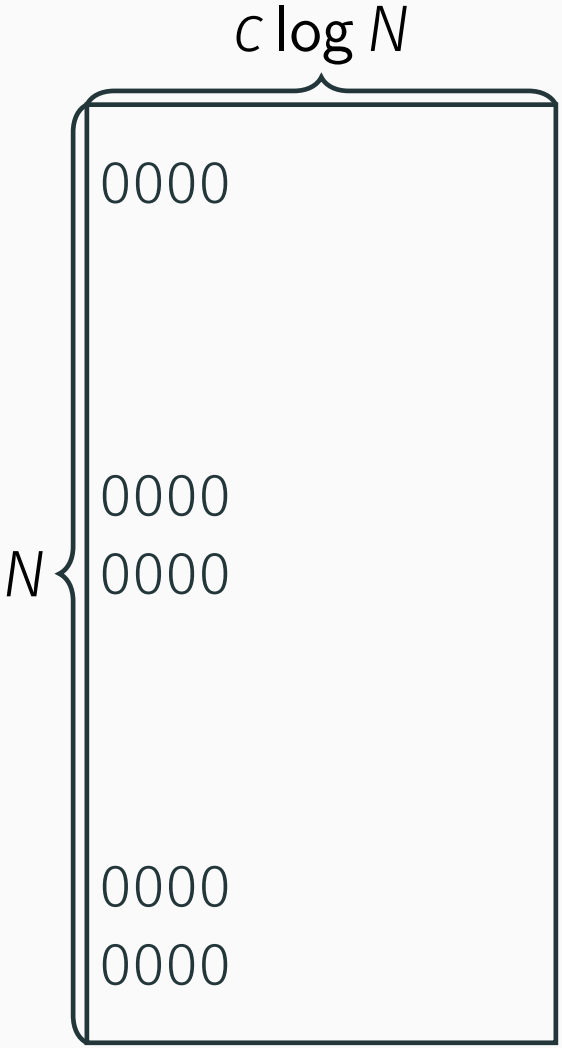

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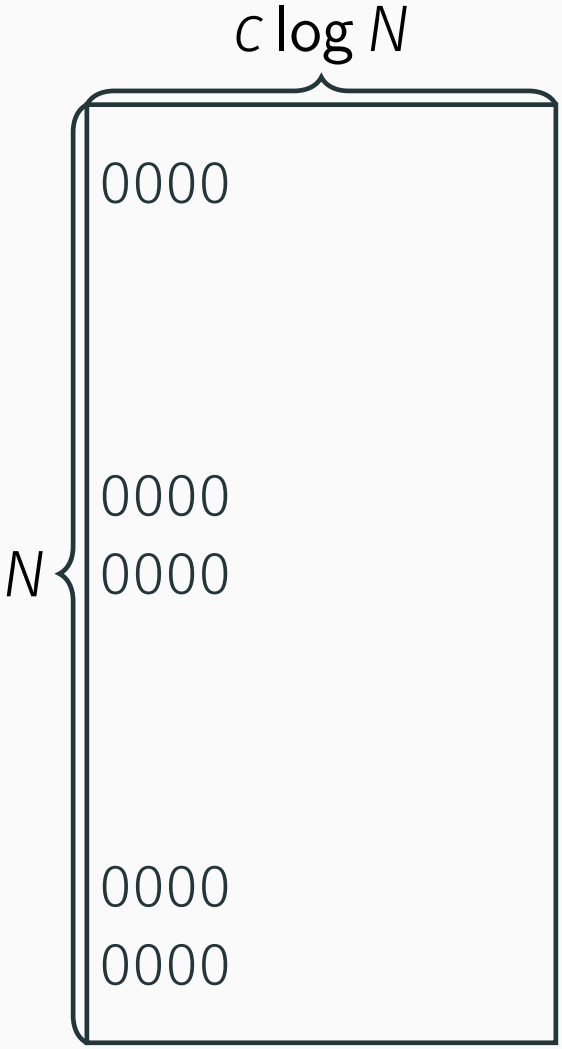

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this structure gives us a subinstance of Online OV in $(c - \varepsilon) \log N$ dimensions!

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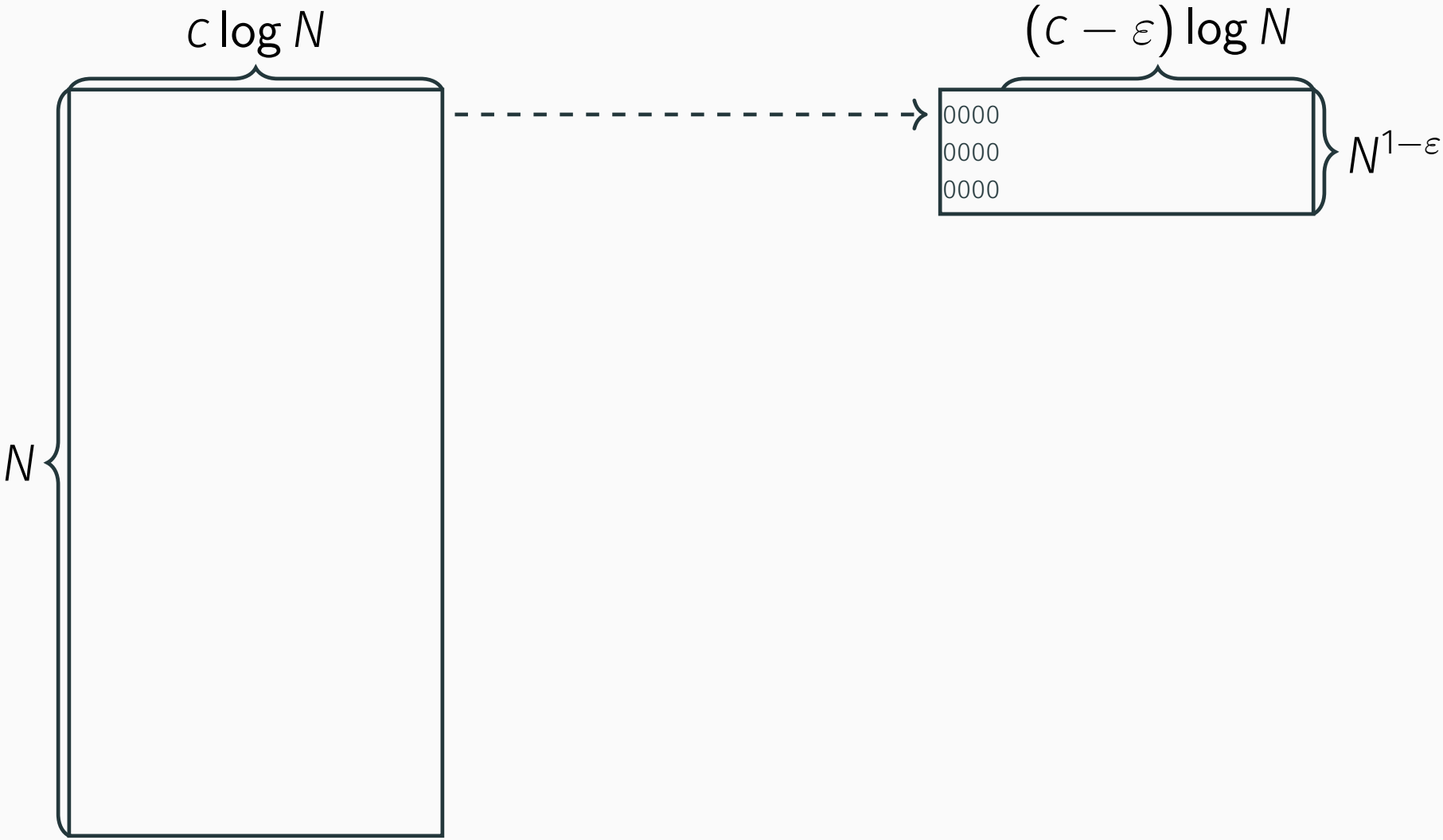

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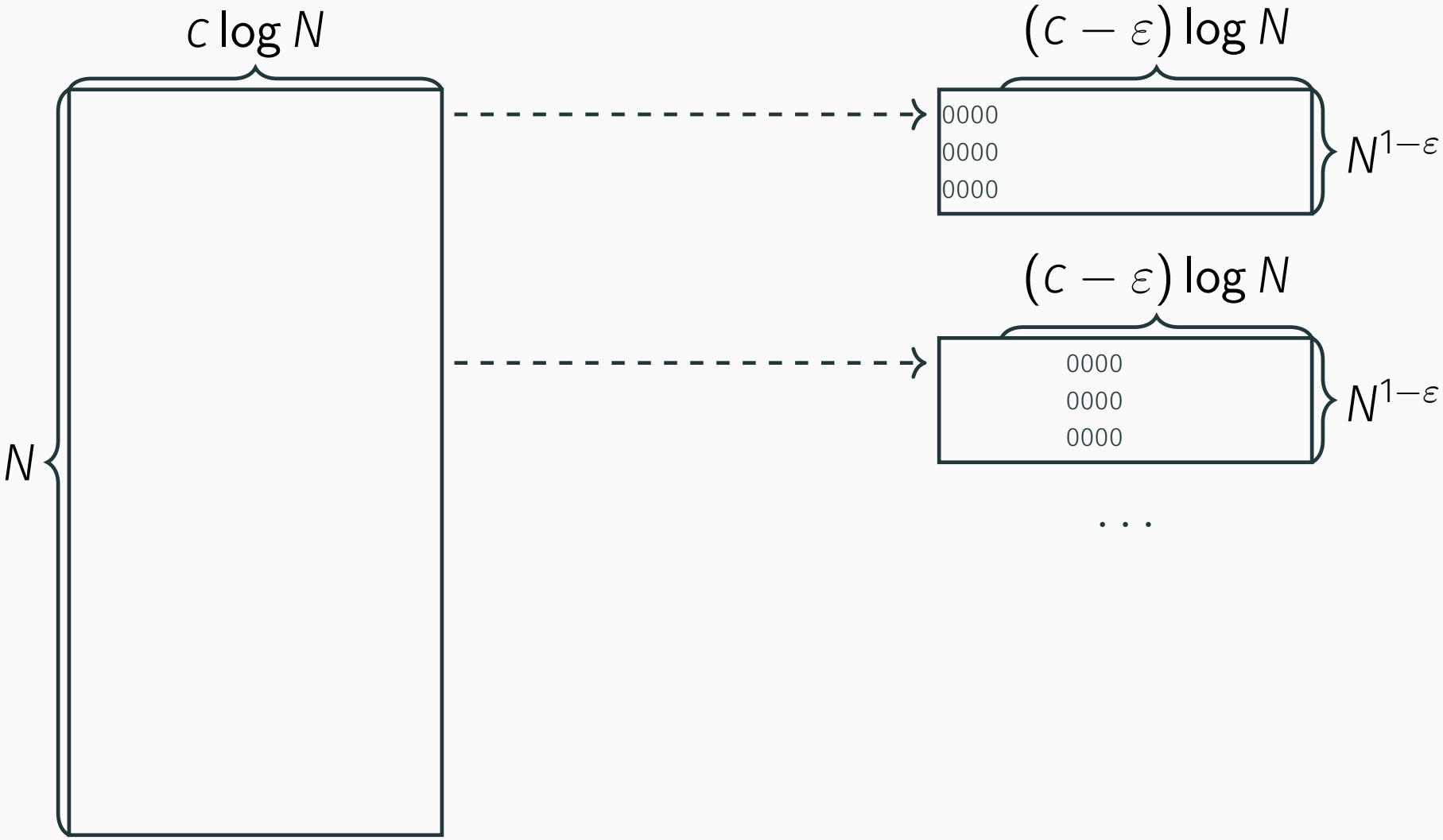

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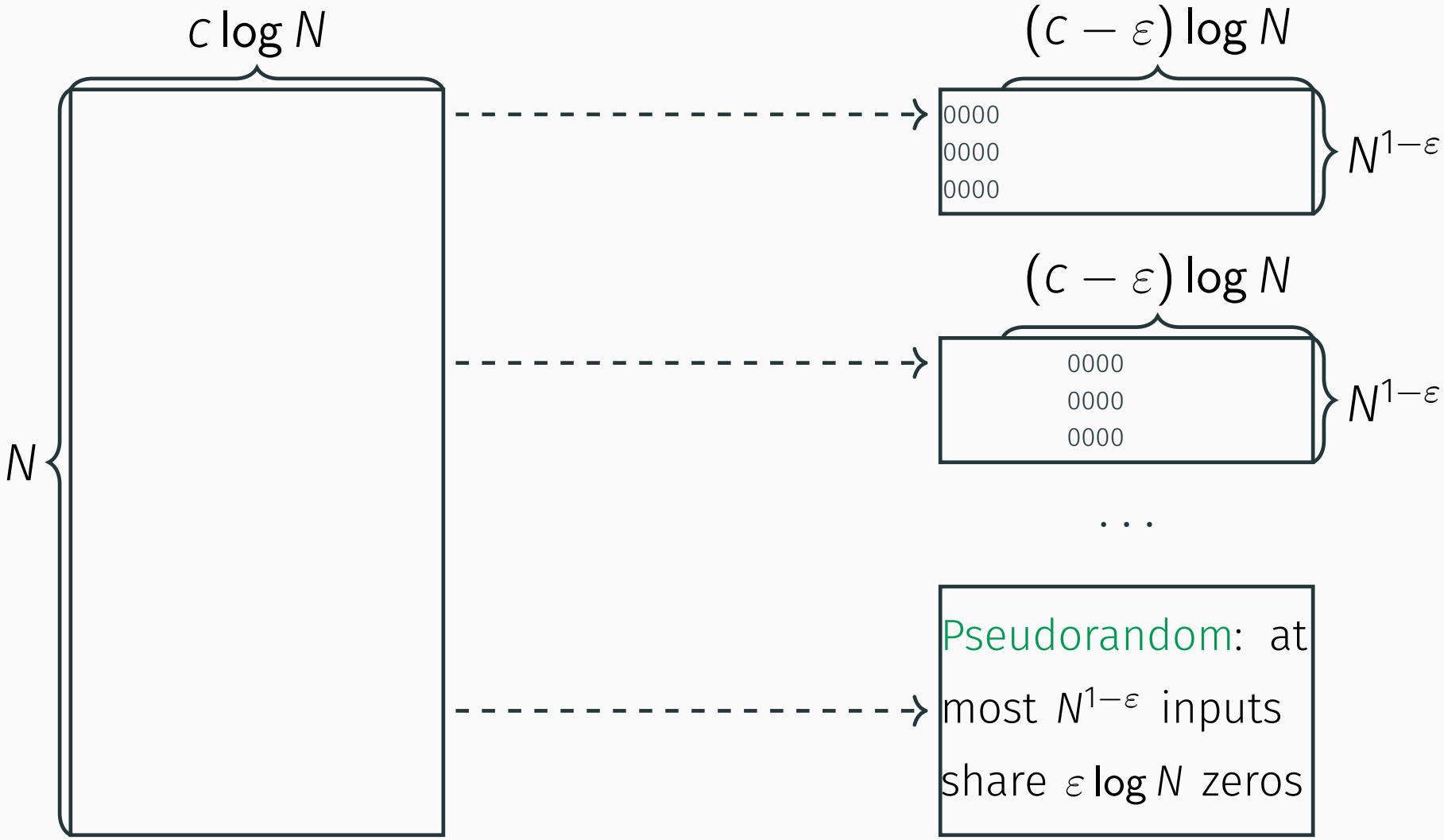

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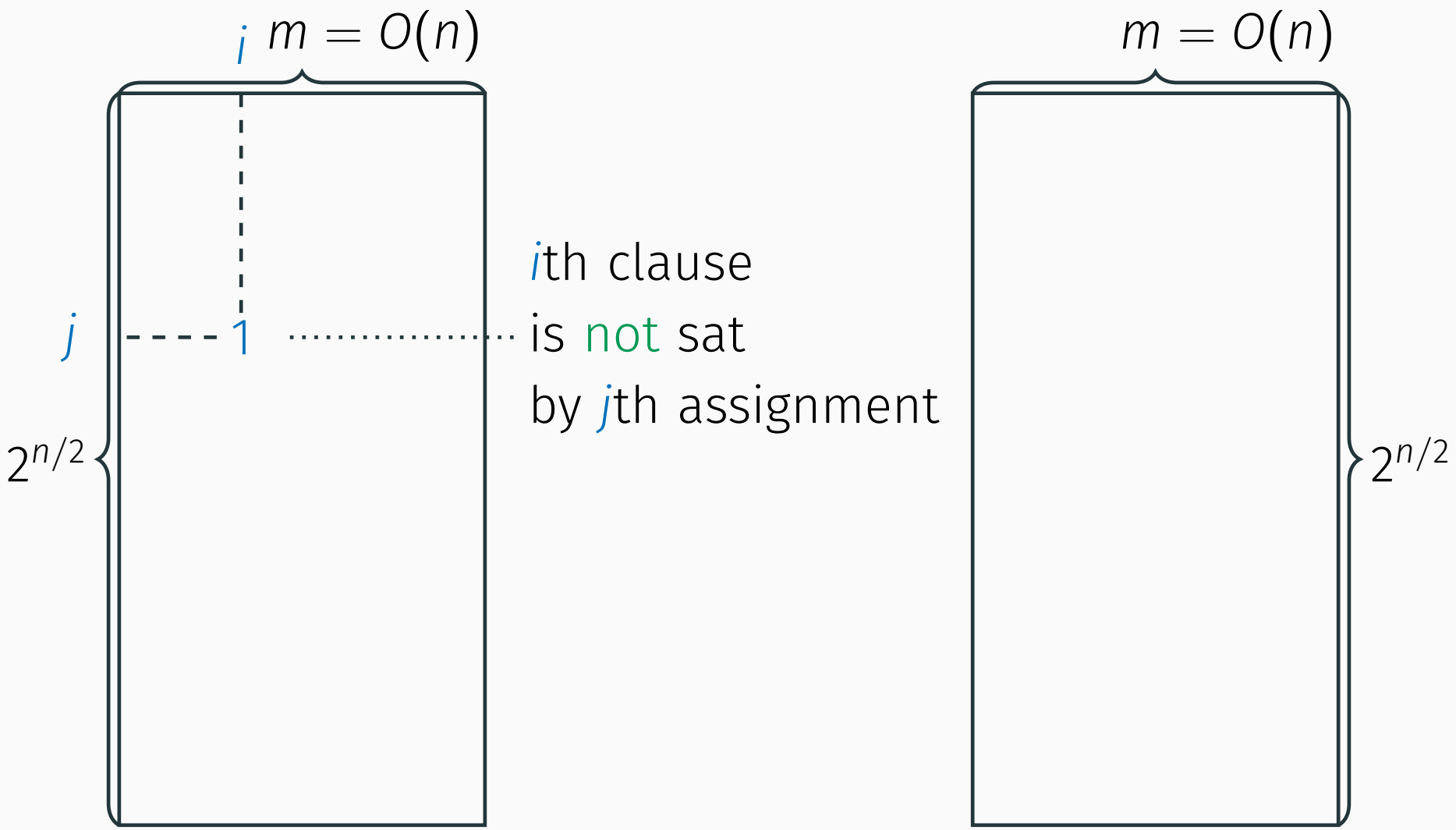
- Holy grail: $S = O(N \cdot d)$, $T \geq N^{\varepsilon}$

SETH Lower Bound



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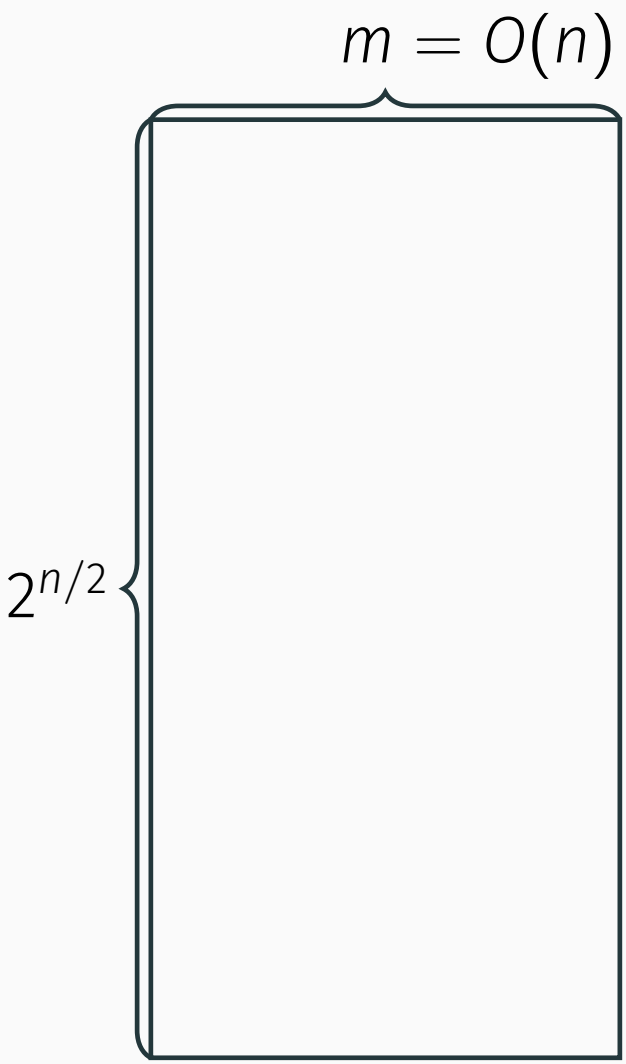

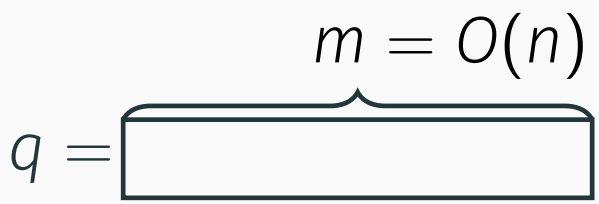

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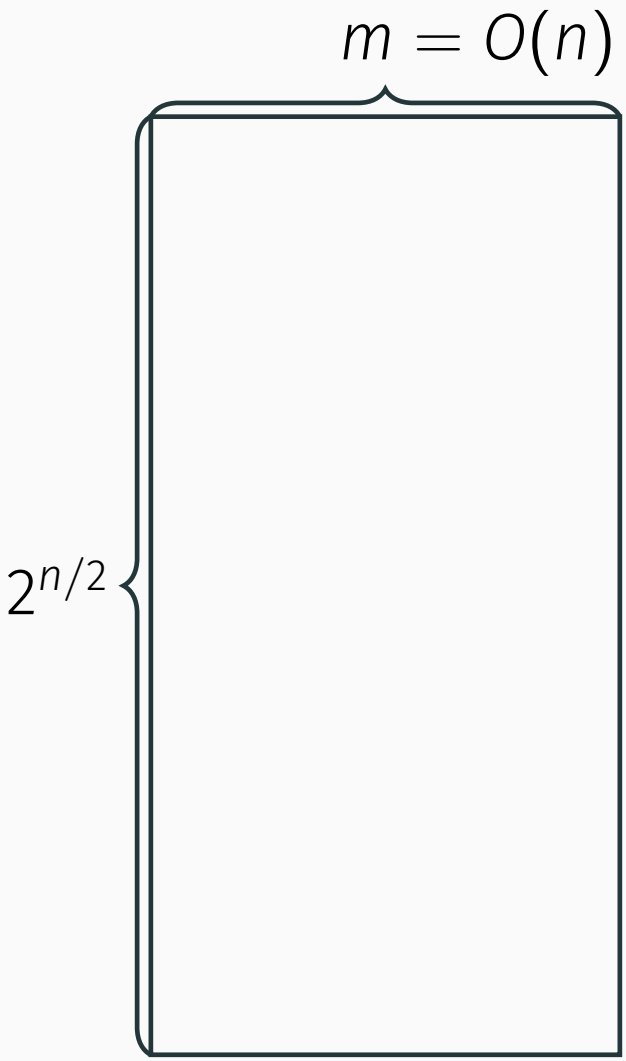

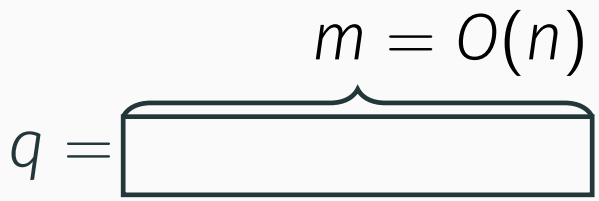

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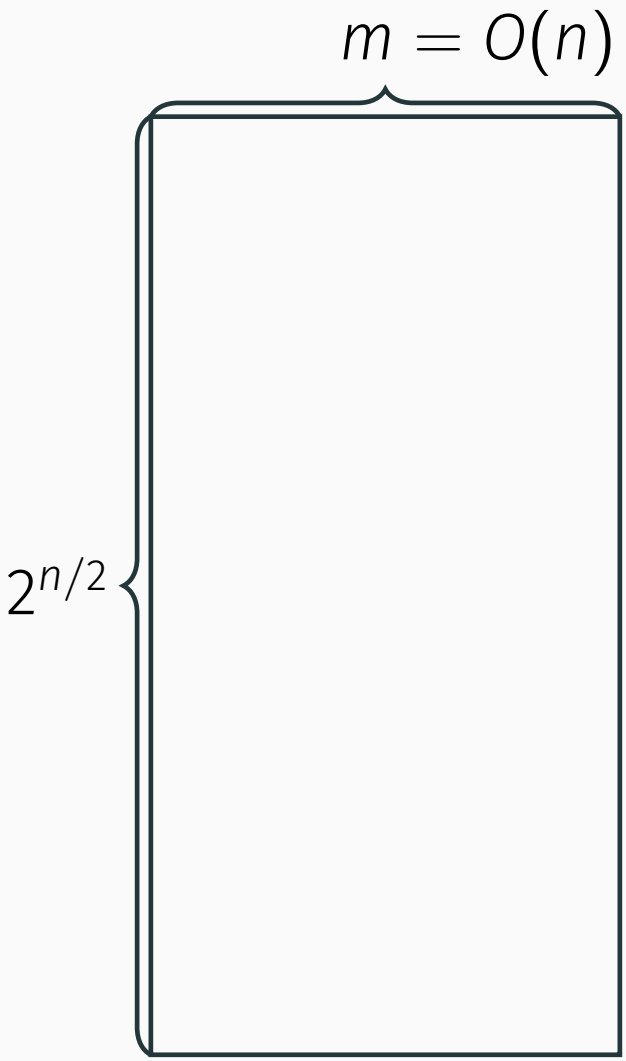

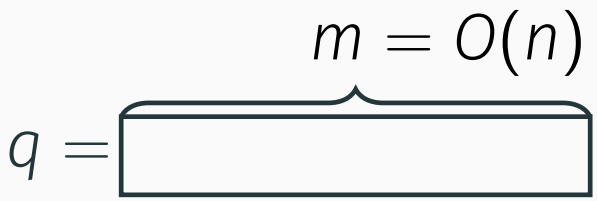

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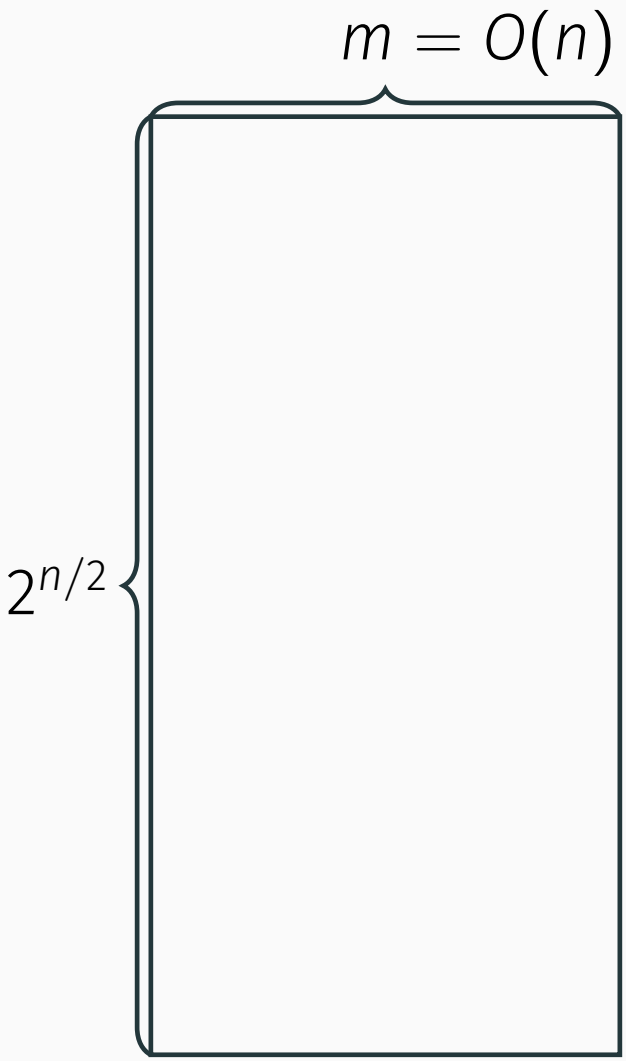
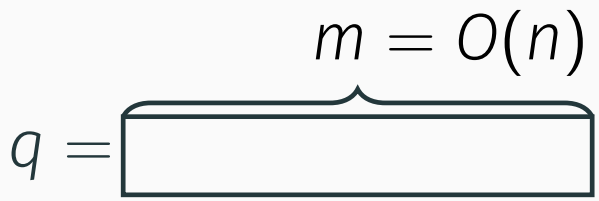
Data structure with $T_p = N^{C}$ and $T = N^{1-\varepsilon}$ would refute SETH

SETH Lower Bound





Data structure with $T_p = N^{2-\varepsilon}$ and $T = N^{1-\varepsilon}$ would refute SETH

Data structure with $T_p = N^C$ and $T = N^{1-\varepsilon}$ would refute SETH

Not a time-space tradeoff!

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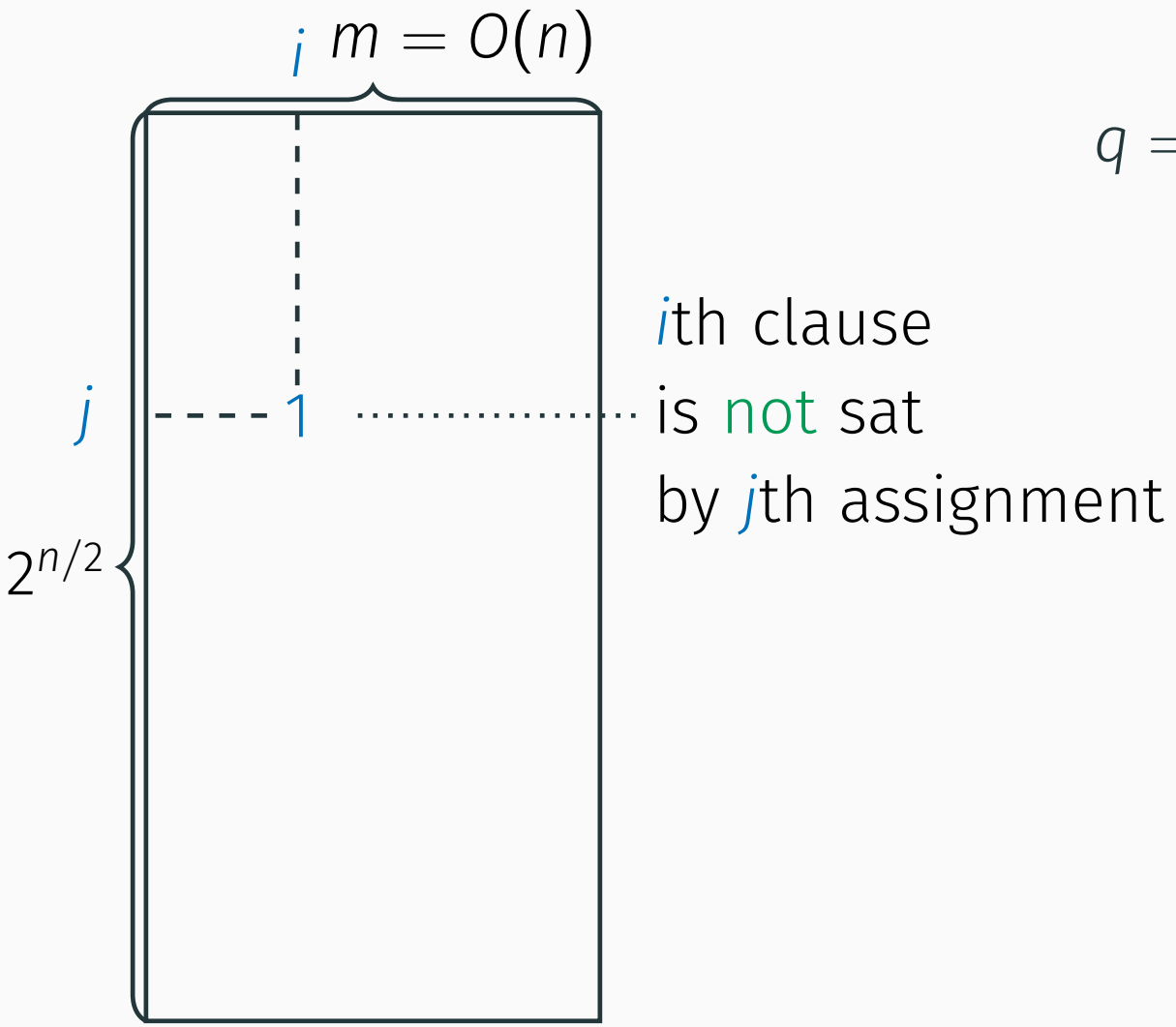
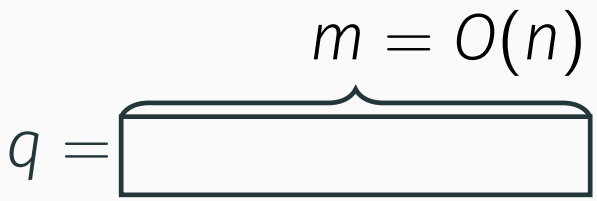
Non-Uniform Strong Exponential Time Hypothesis (NUSETH)

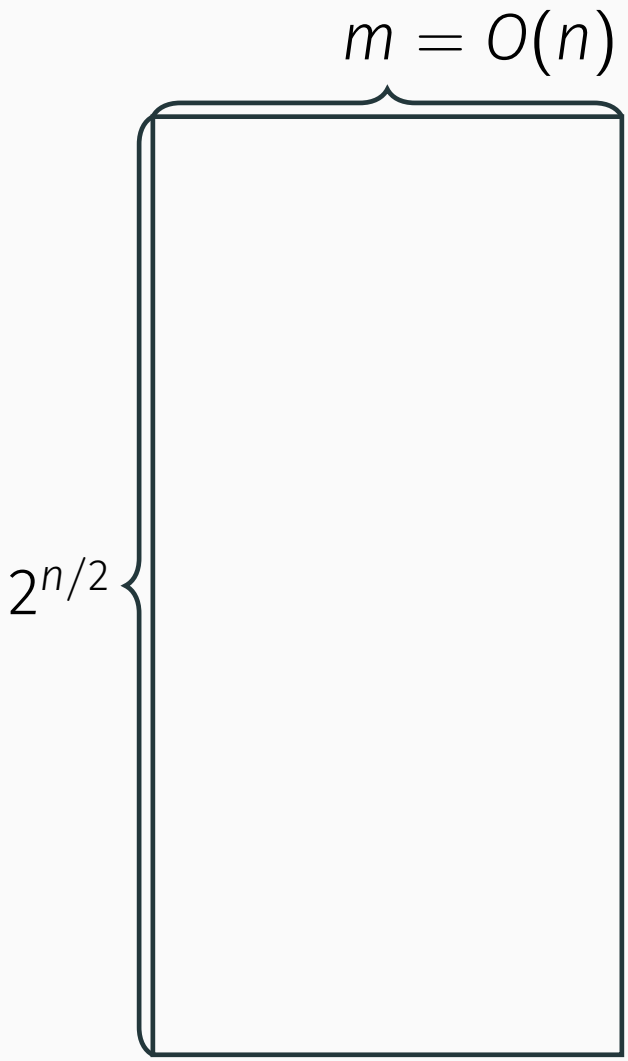
For every $\varepsilon > 0$, there exists k such that no non-uniform algorithm solves k-SAT on formulas with n variables in time $2^{(1-\varepsilon)n}$ (with advice of length $2^{(1-\varepsilon)n}$)

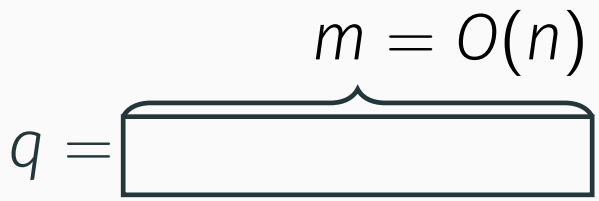
NUSETH Lower Bound





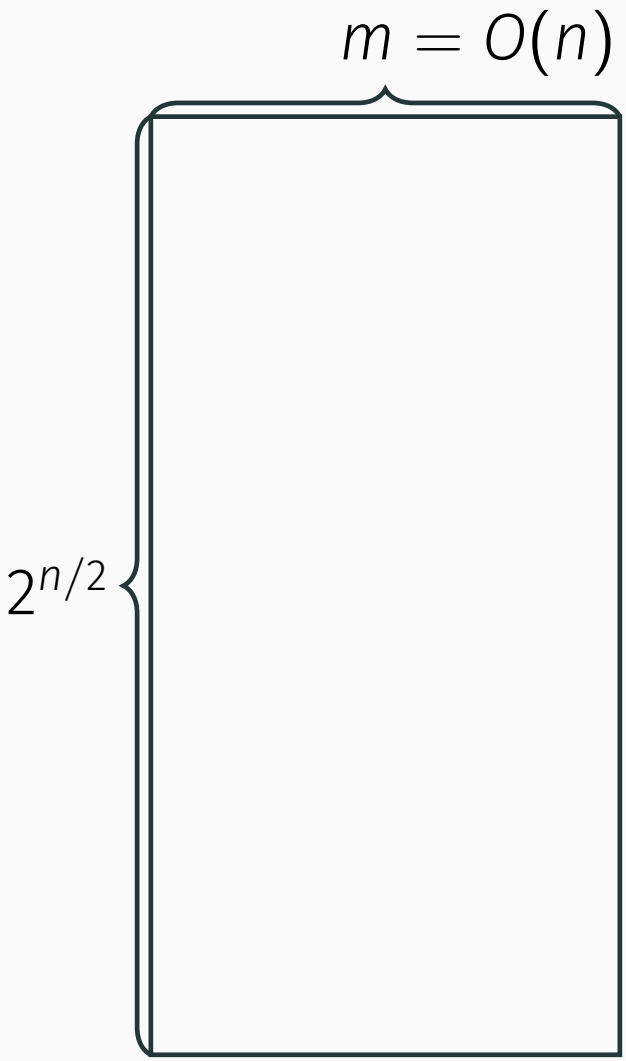
NUSETH Lower Bound

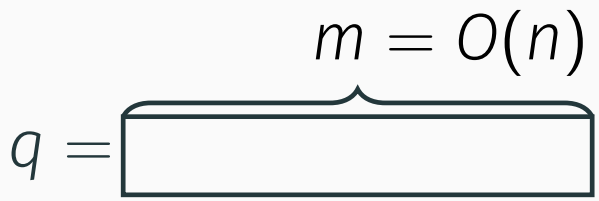




The OnlineOV instance depends on φ, not on n

NUSETH Lower Bound

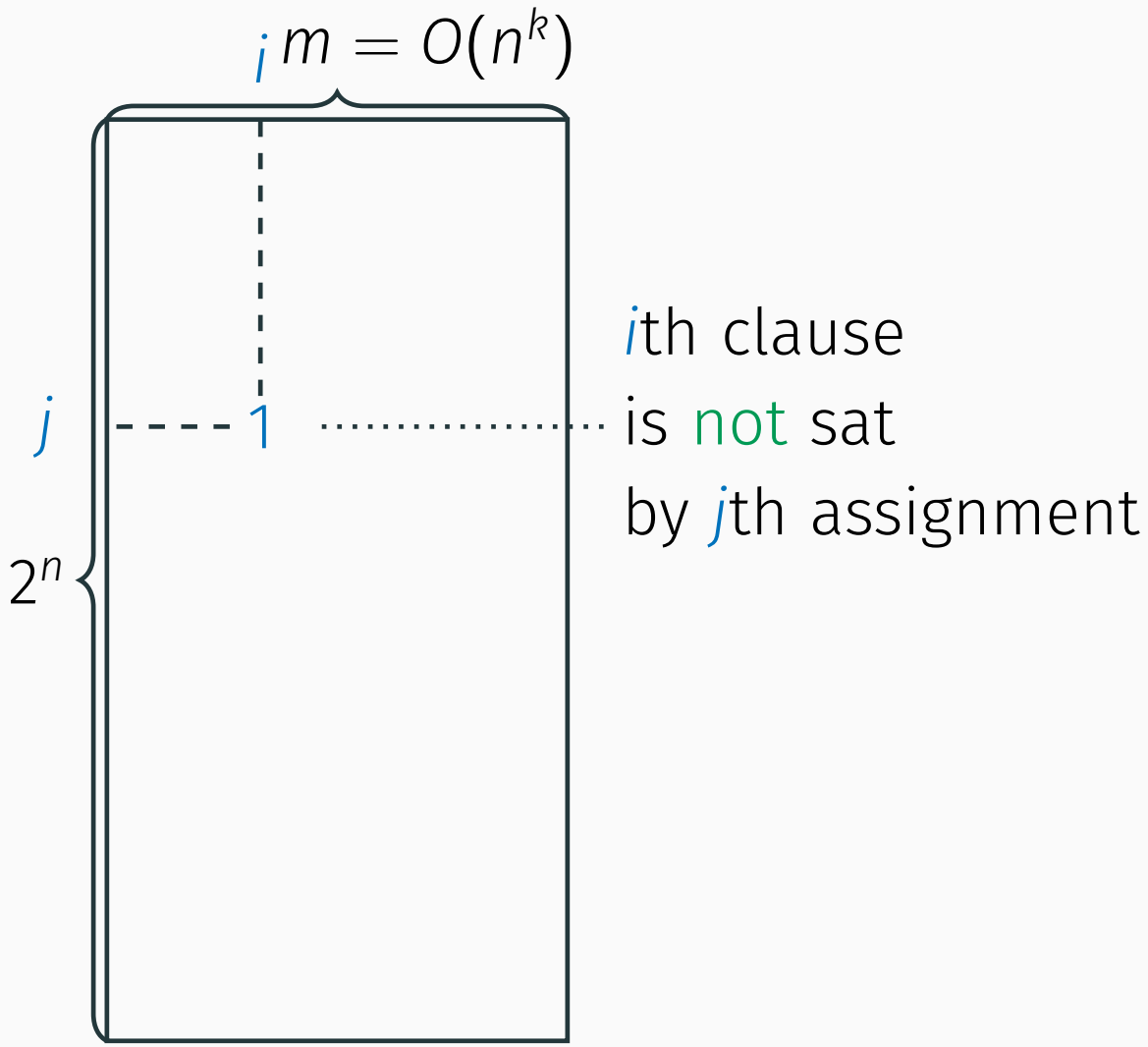




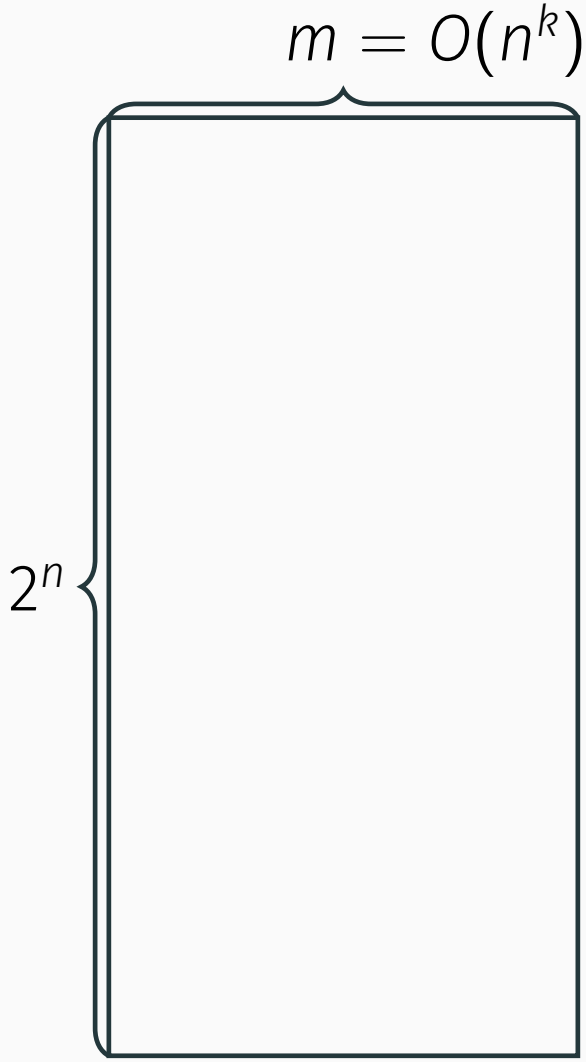
The OnlineOV instance depends on φ, not on n

We need one instance of OnlineOV that solves all φ's

Hardest Instance of OnlineOV



Hardest Instance of OnlineOV

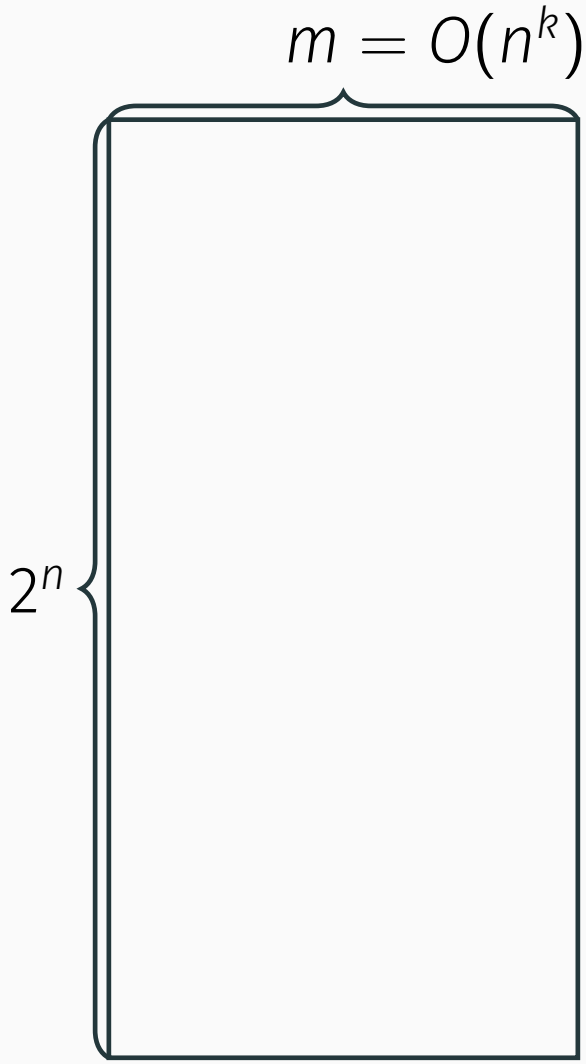


Given k-CNF formula φ, create query q

$q =$ 

Indicator vector of all clauses in φ

Hardest Instance of OnlineOV



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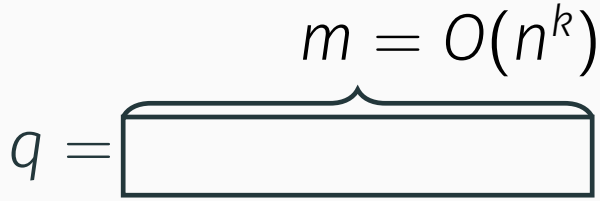

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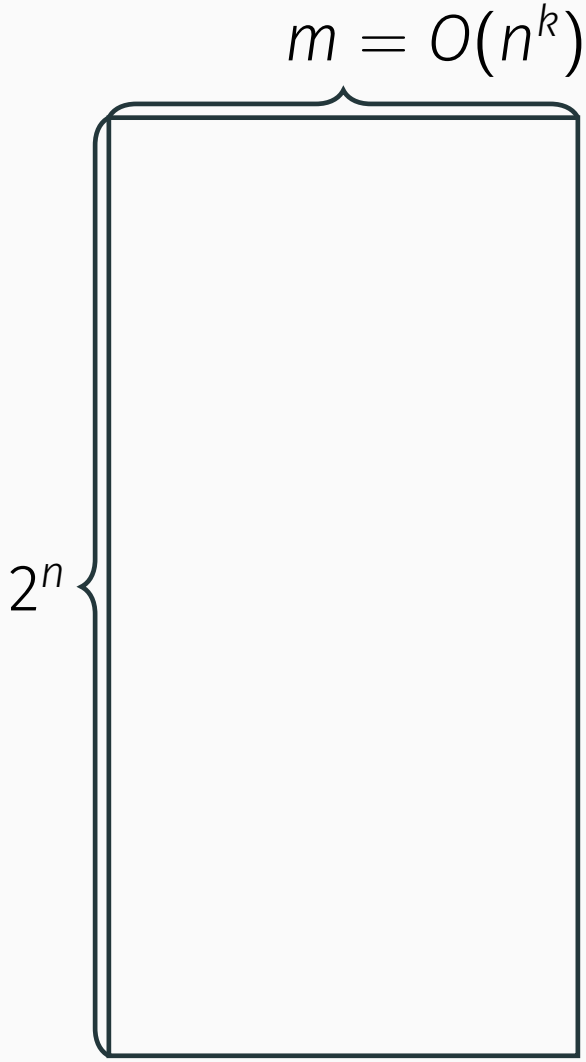

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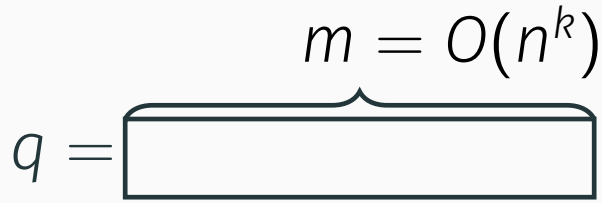

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Use downward self-reducibility to get $2^{n/2}$ formulas over $n/2$ variables.

Generalizations

- Under NUSETH, there is no $(N^c, N^{1-\varepsilon})$-data structure with unbounded preprocessing for OnlineOV in $d = \mathsf{poly}(\log N)$ dimensions. ([AV21]).

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- Lower bounds for PartialMatch, SubsetQuery, ContainmentQuery, OrthogonalRangeSearch, DNFEvaluation, OnlineIP, ApproxNearestNeighbors, ...

Thanks!

Theorem (Upper Bound)

There is deterministic (S, T)-data structure for OnlineOV in $d = c \log N$ dimensions with $S = T_p = N^{1.1}$ and $T = N^{1-\varepsilon}$

Theorem (Lower Bound)

Under NUSETH, there is no $(N^c, N^{1-\varepsilon})$-data structure with unbounded preprocessing for OnlineOV in $d = \mathrm{poly}(\log N)$ dimensions