# Stablity-Preserving, Time-Efficient Mechanism for School Choice in Two Rounds

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# Stable Matching

# Classic Stable Matching Problem

**Problem (Stable Matching)**

*There is a set of $n$ boys and $n$ girls. Each boy has a preference list that is a total order over the girls, and similarly each girl has a preference list that is a total order over the boys.*

**Blocking Pair:** *A boy $b$, and girl $g$, form a blocking pair to an assignment of boys and girls if they both prefer each other over their partners in the assignment.*

**Goal:** *Output a perfect matching of boys and girls with no blocking pairs.*

# Classic Stable Matching

**Problem (Stable Matching Problem)**

**Solution [Gale, Shapley]**

*There exists a mechanism (Differed Accept) that produces a stable matching with the following properites:*

- *Differed Accept runs in polynomial time*
- *Differed Accept produces a girl-optimal stable matching, i.e. each girl gets teh best possible partner she could have gotten in any stable matching*
- *Differed Accpet is startegy-proof (DSIC) for girls* **[Dubins,..]**

# School Choice

## Problem (Stable Matching for Schol Choice)

- *Set of Schools,* $H = \{h_1, ..., h_n\}$
- *Set of Students,* $S = \{s_1, ...s_m\}$
- *Preference list of students over schools* $l(s_i)$ *for all* $s_i \in S$*, where* $l(s_i) = \pi(H \cup \emptyset)$
- *Preference list of schools over students* $l(h_j)$ *for all* $h_j \in H$*, where* $l(h_j) = \pi(S \cup \emptyset)$
- *capacity function* $c : H \to Z$*, such that* $c(h_j)$ *represents the capacity of school* $h_j$*.*

*Updated Blocking Pair:*

- $h_j$ *preferes* $s_i$ *to one of the students assinged to* $h_j$ *(type 1), or*
- $h_j$ *is under-filled and* $h_j$ *prefers* $s_i$ *to* $\emptyset$ *(type 2).*

# Structural Properties of Stable Matchings

**Theorem**

**Rural Hospitals Theorem [R86]:**

- *Over all the stable matchings of the given instance: the set of matched students is the same and the number of students matced to each school is also the same.*
- *Assume that school, h, is not matched to capacity in one stable matching. Then, the set of students matched to h is the same over all stable matchings.*

**Theorem**

*The set of stable matchings characterize a finite distributive lattice.*

# Two Round Setting

We consider a two round setting.

- In Round $\mathcal{R}_1$, mechanism $\mathcal{M}_1$ finds a student-optimal stable matching
- In round $\mathcal{R}_2$, the parameters of the problem change, and we require a mechanism $\mathcal{M}_2$ that returns a stable matching consistent with the new parameters

# Types of Results

- **Type A:** Mechanism $\mathcal{M}_2$ is not allowed to reassingn the school of any students matched by $\mathcal{M}_1$
- **Type B:** Mechanism $\mathcal{M}_2$ is allowed to reassign the school of students matched by $\mathcal{M}_1$, but it must provably minimize ssuch reassignments
- **Type C:** NP-Hardness Results for $\mathcal{M}_2$

# Setting A1

- Let $M$ be the stable matching found by $\mathcal{M}_1$ in $\mathcal{R}_1$.
- Let $S_M$ denote the students who got matched in $\mathcal{M}_1$.
- Let $L = S - S_M$ be the set of students who don't get matched in round $\mathcal{R}_1$

**Theorem**

*There is a polynomial time mechanism $\mathcal{M}_2$ that extends matching $M$ to $M'$ so that $M'$ is stable w.r.t students $S$ and schools $H$. Furthermore $\mathcal{M}_2$ yields the largest matching that can be obtained by the mechanism satisfying these conditions.*

# Mechanism

- For each school $h_j$ find the first student on its preference list such that $s_i \in S_M$ and $s_i$ prefers $h_j$ $S_M$ and $s_i$ prefers $h_j$ to current school. (Barrier($h_j$) $= s_i$)
- For each $s_i \in L$ update their preference lists to only include schools where they are left of the barrier for that school.
- Assign each $s_i \in L$ their favorite school from their preference list
- Return updated matching $M'$

# Proof Idea

- Are any blocking pairs induced?
- What about students who didn't get matched in round $\mathcal{R}_2$
- Is it incentive compatible??

# Incentive Compatibility

**Example**

$$\begin{aligned} S_1 &: \quad H_1 \quad H_2 \\ S_2 &: \quad H_1 \quad H_2 \\ H_1 &: \quad S_1 \quad S_2 \\ H_2 &: \quad S_2 \quad S_1 \end{aligned}$$

Each school has capacity 1 in round $\mathcal{R}_1$

- Truthful reporting will result in $M = M' = (S_1, H_1), (S_2, H_2)$
- If $S_2$ instead misreports her preference list as $(H_1, \emptyset)$ then $M = (S_1, H_1)$, and $M' = (S_1, H_1), (S_2, H_1)$.
- $S_2$ does better by cheating.

# Setting A2

This setting follows from Setting A1

- Let $N$, be a set of new students who also arrive in round $\mathcal{R}_2$.
- $MIN_N MAX_L$ asks for a stable extension in round $\mathcal{R}_2$ that minimizes the number of students who get matched in $N$ and subject to that maximize the number of students matched from $L$

**Theorem**

*There is a polynomial time mechanism $\mathcal{M}_2$ for:*

- $MIN_N MAX_L$
- $MAX_N MAX_L$
- $MAX_{N \cup L}$

# Proof Sketch $MIN_N MAX_L$

- $s_i \in N$ who form blocking pairs with schools must be matched
- Let $S_{NL}$ be the students in $N$ who don't form blocking pairs, they won't be in any matching
- Consider the barriers for schools defined by students in $S_M$ and $S_{NL}$, and set the barrier to be the stricter of the two.
- update the preference lists of all students in $L, N - S_{NL}$ to include only schools where they lie to the left of the barrier.
- Matching these students to their favorite school results in a stable extentions that minimizes the number of students in $N$, and subject to that maximizes $L$
- $MAX_N MAX_L, MAX_{N \cup L}$ can be done similarly

# Some NP Hardness for A2

**Theorem**

*The following problems are NP-Hard:*

- $MAX_L MIN_N$
- $MAX_N MIN_L$
- *Choose $k$ students from $N$, such that it will maximize the number of students matched from $L$*

## Setting B1

- In round $\mathcal{R}_2$ a set of new schools $H'$ arrive, and original schools can increase their capacity.
- Students can now move, but we want to minize the number of students who are re-allocated in round $\mathcal{R}_2$.

**Theorem**

*There is a polynomial time mechansim $\mathcal{M}_2$ that finds a minimum stable re-allocation with respect to round $\mathcal{R}_1$ matching $M$, students $S$, and schools $H \cup H'$*

# Setting B1 GS-counterexample

**Example**

$$\begin{array}{lll} S_1: & H_2 & H_1 \\ S_2: & H_1 & H_2 \\ H_1: & S_1 & S_2 \\ H_2: & S_2 & S_1 \end{array}$$

In round $\mathcal{R}_1$ school $H_1$ has one seat and and school $H_2$ arrives in round $\mathcal{R}_2$.

- Round $\mathcal{R}_1$ matching is just $(S_1, H_1)$
- Running Gale-Shapley in round $\mathcal{R}_2$ results in $(S_1, H_2), (S_2, H_1)$ requiring one re-allocation.
- However there is a stable matching $(S_1, H_1), (S_2, H_2)$ that requires no re-allocations.

# Structural Properties of MSR

**Lemma**

*Each student weakly improves in any minimum stable re-allocation*

**Lemma**

*All minimum stable re-allocations move the same set of students,* $S_R$

## Proof Sketch

- Let there be two $MSR$, such that $s_i$ is moved in one and not in the other, i.e.$M(s_i) = M'(s_i) \neq M''(s_i)$.
- Then the following are possible cases for $s_i$:
  1. $S_1 = \{s_i \in S_M | M(s_i) = M'(s_i) \neq M''(s_i)\}$
  2. $S_2 = \{s_i \in S_M | M(s_i) = M''(s_i) \neq M'(s_i)\}$
  3. $S_3 = \{s_i \in S_M | M(s_i) \neq M'(s_i), M(s_i) \neq M''(s_i)\}$
  4. $S_4 = \{s_i \in S_M | M(s_i) = M'(s_i) = M''(s_i)\}$
- Consider the matching $M_L = M' \wedge M''$ where each student goes to the school she prefers less.
- By the first lemma this will send all $s_i$ in $S_1, S_2$ and $S_4$ to their original schools
- $M_L$ requires fewer re-allocations, a contradiction.

# MSR Lattice

**Theorem**

*The set of minimum stable re-allocations form a sub-lattice of the stable matching lattice.*

- You can divide students into two groups moved, fixed
- Since the students who move are fixed, you can define a smaller stable matching instance over these students
- adding the matching restricted to the fixed studennts will give a minimum stable re-allocation

# Mechansim for adding School

- While there exists a school with a vacant seat, and a student who prefers that school to its current match, match the school and student
- the above mechanism terminates and returns a school-optimal minimal stable re-allocation.
  (Proof by Induction)
- To get a student-optimal minimum stable re-allocation, find the moving students from running school-optimal mechanism, then construct special stable matching instance over moving students and find a school-optimal matching there.
- Not incentive-compatible!!!!

# Setting B2

- In round $\mathcal{R}_2$ a set of new students arrive, the capacity of schools doesn't change

**Theorem**

*There is a polynomial mechanism $\mathcal{M}_2$ that finds a minimum stable re-allocation with respect to round $\mathcal{R}_1$ maching $M$, students $S \cup N$, and schools $H$.*

# Open Problems

- Incentive Compatible Mechanisms??
- Approximation Algorithms for NP-Hard Problems??

**Thank You!**